PH 561. 4

Reg. No. | | | | | | |

St. Aloysius College (Autonomous), Mangaluru

Semester IV - P. G. Examination - M. Sc. Mathematics

May 2023

Measure Theory and Integration

Time : 3 Hours

Max. Marks : 70

Answer any FIVE full questions.

1. (a) Show that for any $A \subseteq \mathbb{R}$ with $m^*(A) < +\infty$, and $\epsilon > 0$, there is an open set O containing A such that $m^*(O) < m^*(A) + \epsilon$.

   (b) Prove that outer measure of a compact interval is its length.

   (c) Show that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ for any set B. (4+7+3)

2. (a) State and prove the countable additive property of the Lebesgue measure.

   (b) If $\{E_i\}$ is a sequence of Lebesgue measurable sets such that $E_1 \subseteq E_2 \subseteq \cdots$, then prove that $m(\lim E_i) = \lim m(E_i)$.

   (c) Show that every non-empty open set has positive measure. (7+5+2)

3. (a) If E is a Lebesgue measurable set, then show that $V = \{f : E \to \mathbb{R} \ : \ f \text{ is measurable}\}$ is a vector space over $\mathbb{R}$. Further, show that V is closed under the multiplication of functions.

   (b) If f is a measurable function and B is a Borel set, show that $f^{-1}(B)$ is a measurable set.

   (c) Show that a continuous real function f is measurable. (4+6+4)

4. (a) Define the Lebesgue integral of a non-negative Lebesgue measurable function f. Prove that $f = 0$ a. e. if $\int f \, dx = 0$.

   (b) Let f be a non-negative measurable function. Prove that there exists a increasing sequence $\{\phi_n\}$ of measurable simple functions such that $\phi_n \to f$.

   (c) If f and g are non-negative measurable functions and if c is a non-negative real number, then show that $\int (cf + g) dx = c \int f dx + \int g dx$. (4+5+5)

5. Let $f : [a, \ b] \to \mathbb{R}$ be a bounded function.

   (a) If f is Riemann integrable, then prove that f is Lebesgue integrable.

   (b) If f is an integrable function, then prove that $|\int f dx| \leq \int |f| dx$. Also discuss the case of equality. (8+6)

6. (a) Define a measure and a measure space. Show that $L^p(X, \ \mu)$, $1 \leq p \leq \infty$, is a vector space over $\mathbb{R}$.

   (b) Define the notion of a convex function. Prove that a convex function defined on an open interval is continuous. (6+8)

7. State and prove Holder's inequality. Show that equality occurs if and only if $sf^p + tg^q = 0$ a.e for some constants s and t, not both zero. (14)

8. State and prove Hahn's lemma and Hahn's decomposition theorem. (14)

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Mangaluru

Semester IV – P.G. Examination – M.Sc. Mathematics

May /June - 2023

COMPLEX ANALYSIS - II

Time: 3 hrs. Max Marks: 70

Answer any FIVE FULL questions. (14x5=70)

1. a) Define a simply connected region. Give two examples.
   b) Prove that a region Ω is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω.

   (2+12)

2. a) State and prove residue theorem.
   b) i) show that $\int_{|z|=2} \frac{dz}{(z-3)(z^5-1)} = \frac{-i\pi}{2}$

   ii) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $a > b > 0$. (8+6)

3. a) State and prove argument principle.
   b) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?
   c) Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5} dx$ (7+3+4)

4. a) Show that arithmetic mean of a harmonic function $u(z)$ over a concentric circles $|z| = r$ is a linear function of $\log r$. Further, prove that, if $u(z)$ is a harmonic in a whole disc, then arithmetic mean is constant.
   b) State and prove the maximum principle for harmonic functions.

   (7+7)

5. a) Prove the Poisson formula for the function $u(z)$ harmonic in $|z| < R$ and continuous on $|z| < R$, for each with $|a| \le R$, $u(a) = \frac{1}{2\pi}\int_{|z|=R} \frac{R^2-|a|^2}{|z-a|^2} u(z)\, d\theta$.
   b) State and prove reflection principle for harmonic functions. (8+6)

6. a) If the function $f_n(z)$ are analytic and $\neq 0$ in a region Ω and if $f_n(z)$ converges to $f(z)$ uniformly on every compact subset Ω, then show that $f(z)$ is either identically zero or never equal to zero in Ω.
   b) Show that $\lim_{n\to\infty}\left(1+\frac{z}{n}\right)^n = e^z$ uniformly on all compact subset of the complex plane. (7+7)

7. a) Obtain the Laurent's development of an analytic function in an annulus region. Further prove that Laurent development is unique.
   b) Show that $\pi \cot \pi z = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{(z^2-n^2)}$. (9+5)

8. a) Define convergence and absolute convergence of $\prod_{n=1}^{\infty}(1+a_n), 1+a_n \neq 0, \forall n.$ Derive necessary and sufficient conditions for
   i) Convergence and
   ii) Absolute convergence of $\prod_{n=1}^{\infty}(1+a_n)$.
   b) State and prove Poisson-Jensen formula for entire functions.
   c) Show that a meromorphic function is the ratio of two entire functions.

   (7+4+3)

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Semester IV– P.G. Examination – M.Sc. Mathematics

May/June –2023

FUNCTIONAL ANALYSIS

Time: 3 hrs **Max Marks: 70**

Answer any FIVE FULL questions from the following: $(14 \times 5 = 70)$

1. a) Let X be a complete metric space and $\{F_n\}$ be a decreasing sequence of nonempty closed sets in X such that $d(F_n) \to 0$. Then prove that $\cap_{n=1}^{\infty} F_n$ contains exactly one point.

   b) State and prove the Baire's category theorem. **(6+8)**

2. Let p be a real number such that $p \geq 1$. Prove that the linear space l_p of all sequences of $x = (x_1, x_2, \ldots)$ of scalars such that $\sum_{i=1}^{\infty}|x_i|^p < \infty$, forms a Banach space with respect to the norm given by $\|x\|_p = (\sum_{i=1}^{\infty}|x_i|^p)^{\frac{1}{p}}$.

   (14)

3. a) Let N and N' be normed linear spaces. Prove that the following are equivalent for a linear transformation $T: N \to N'$

   i) T is continuous.

   ii) T is continuous at the origin.

   iii) There exists a real number $K \geq 0$, such that $\|T(x)\| \leq K\|x\|$ for all $x \in N$.

   iv) If $S = \{x \in N: \|x\| \leq 1\}$ then $T(S)$ is bounded set in N'.

   b) When do we say that two norms in a linear space are equivalent? If L is a linear space made into a normed linear space by $\|.\|$ and $\|.\|'$ then show that these two norms are equivalent if and only if there exist positive reals K_1 and K_2 such that $K_1\|x\| \leq \|x\|' \leq K_2\|x\|$ for all $x \in L$.

   (9+5)

4. a) State and prove the closed graph theorem for a linear transformation between Banach spaces.

   b) Show that a normed linear space N can be imbedded in its second conjugate space N^{**}. **(10+4)**

5. If T is a continuous linear transformation of a Banach space B onto a Banach space B' then prove that T is an open map. **(14)**

6. a) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$.

   b) If M and N are closed linear subspace of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed.

   c) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$. **(5+6+3)**

Contd...2

7. a) For a finite orthonormal set $\{e_1, e_2, \dots, e_n\}$ in a Hilbert space H and $x \in H$ show that $x - \sum_{i=1}^{n} \langle x, e_i \rangle e_i \perp e_j$, for each $j, 1 \le j \le n$.

b) Let $\{e_i\}_{i \in I}$ be an orthonormal set in a Hilbert space H. Show that for any vector $x \in H$, the set $\{e_i : \langle x, e_i \rangle \neq 0\}$ is at most countable and prove that $\sum_{i \in I} |\langle x, e_i \rangle|^2 \le \|x\|^2$.

(4+10)

8. a) Let H be a Hilbert space, prove that for each $T \in B(H)$ there exists a unique operator $T^* \in B(H)$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x, y \in H$.

b) Define a Unitary operator on a Hilbert space H. Show that a unitary operator T is an isometric isomorphism of H into itself.

(8+6)

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Semester IV – P.G. Examination – M.Sc. Mathematics

May /June 2023

PARTIAL DIFFERENTIAL EQUATIONS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following: **(14x5=70)**

1.a) Prove that a necessary and sufficient condition for a differential equation $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ to be integrable is that $X.\text{curl } X = 0$.

b) Find the integral curves of the equations $\frac{dx}{2xz} = \frac{dy}{2yz} = \frac{dz}{z^2 - x^2 - y^2}$.

c) Test for integrability of $yzdx + 2xzdy - 3xydz = 0$ and find its primitive.

(7+3+4)

2.a) Prove that there exists a relation of the form $F(u,v) = 0$ not involving x or y explicitly between two functions $u(x,y)$ and $v(x,y)$ if and only if $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

b) Find the orthogonal trajectories on the cylinder $y^2 = 2z$ on the curve in which it is cut by the system of planes $x + z = c$, where c is a constant. **(7+7)**

3.a) Solve $yz(1 + 4xz)dx - xz(1 + 2zx)dy - xydz = 0$.

b) Find the equation of the integral surface of the partial differential equation $x(y^2 + z)z_x - y(x^2 + z)z_y = (x^2 - y^2)z$ which contains the straight line $x + y = 0, z = 1$.

(7+7)

4.a) Find the characteristic of the equation $z = p^2 - q^2$ and determine the integral surface which passes through the circle $4z + x^2 = 0,\ y = 0$.

b) Show that the equation $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution. **(7+7)**

5.a) Find a complete integral of the equation $(p^2 + q^2)x = pz$ and hence derive the equation of an integral surface of which passes through the curve $x = 0,\ z^2 = 4y$.

b) Find the complete integral of $z^2 = pqxy$ using Charpit's method. **(7+7)**

6.a) Solve $\left(D^2 - 3DD' + 2D'^2\right)z = (4x + 2)\ e^{x+2y}$.

b) Find the particular integral of $\left(D^2 + DD' + 2D'^2\right)z = x^2y$.

c) Solve $\left(D^2 + 4DD' + 2D'^2\right)z = \sin(x + 2y)$. **(7+4+3)**

7.a) Classify the equation $(1 + x^2)u_{xx} + (1 + y^2)u_{yy} + xu_x + yu_y = 0$ and reduce it to canonical form.

b) Reduce the equation $u_{xx} + 2u_{xy} + u_{yy} = 0$ to canonical form.

(9+5)

8.a) Solve $u_{tt} - 4u_{xx} = 0,\ 0 \le x \le 1,\ t > 0$

Satisfying the initial conditions $u(x,0) = x - x^2$

$u_t(x,0) = \sin \pi x \quad 0 \le x \le 1$

Boundary conditions $u(0,t) = 0 \quad u(1,t) = 0 \quad t \ge 0$.

b) Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi,\ t \ge 0$ subject to the conditions

a) T remains finite as $t \to \infty$

b) $T = 0$ if $x = 0$ and $x = \pi$ for all t

c) At $t = 0$ $T = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$

(7+7)

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