PH 561.1 Reg. No:

St Aloysius College (Autonomous)
Mangaluru
Semester I – P.G. Examination- M.Sc. Mathematics
November/December - 2023

ALGEBRA I

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following. **(14x5=70)**

1. a) Let H and K be subgroups of a group G and let $\varphi: H \times K \to G$ be defined by $\varphi(h,k) = hk, \forall (h,k) \in H \times K$. Then prove the following:
   i. φ is one-one if and only if $H \cap K = \{e\}$.
   ii. If H and K are normal subgroups of G such that $H \cap K = \{e\}$ and $HK = G$, then $G \cong H \times K$.

   b) Prove that a subset H of $\mathbb{Z}$ is a subgroup if and only if $H = a\mathbb{Z}$ for some $a \in \mathbb{Z}$.
   (6+8)

2. a) State and prove the Lagrange's theorem.
   b) Define a cyclic group. Prove that every subgroup of a cyclic group is cyclic.
   (8+6)

3. a) Prove the following:
   i. Let G be a finite group and let $a \in G$. Then $o(a) \mid o(G)$.
   ii. Every group of prime order is cyclic.

   b) Define a subgroup of a group G. If G is a group and H is a nonempty subset of G then prove that H is a subgroup of G if and only if $ab^{-1} \in H$, $\forall a, b \in H$.
   c) Define center of a group. Show that center of a group G is a subgroup of G.
   (5+5+4)

4. a) State and prove second Sylow theorem for a finite group.
   b) Define a p-group. If p is a prime number, then prove that any group of order p^2 is cyclic or isomorphic to a product of two cyclic groups each of order p.
   (8+6)

5. a) Let G be a finite group and H be a subgroup of G of index $m > 1$ such that $o(G) \nmid m!$. Then prove that G has a non-trivial normal subgroup.
   b) Define orbit of an element. Let G be a finite group, S be a nonempty set such that G acts on S, and let $s \in S$. Then prove that the order of orbit of s is the index of the stabilizer of s in G, i.e., $|O_s| = [G : G_s]$. Hence show that $o(G) = o(O_s)\, o(G_s)$.
   (6+8)

Contd...2

6. a) Determine all possible class equations of groups of order 21 and 8.

   b) Show that every group of order 15 is simple.

(8+6)

7. a) Define an integral domain. Prove that every integral domain can be embedded in a field.

   b) Prove that, an ideal P of a ring R is a prime ideal if and only if $R/_P$ is an integral domain.

(8+6)

8. a) Prove that every finite integral domain is a field.

   b) Prove that a commutative ring R is a field if and only if the only ideals of R are (0) and itself.

   c) Let R be a ring with identity. Then prove that the set of all units of R is a group with respect to multiplication in R.

(5+5+4)

****

PH 562.1

Reg No : ☐☐☐☐☐☐☐

ST ALOYSIUS COLLEGE (AUTONOMOUS) MANGALURU

Semester I - P.G. Examination - M.Sc. Mathematics

November/December - 2023

LINEAR ALGEBRA I

Time : 3 Hours | Max. Marks : 70

Answer FIVE FULL questions | (14x5=70)

1. a) Define elementary matrices. Prove that elementary matrices are invertible. 6

   b) i) Does the equation $AB - BA = I$ have solution in real $n \times n$ matrices $A$and B? justify 8

   ii) Let A be an$n \times n$ matrix with integer entries. Prove that A is invertible, and that A^{-1} has integer entries if and only if $\det A = \pm 1$.

2. a) Prove that any $m \times n$ matrix can be reduced to row echelon form by using row operations. 7

   b) Let $A = (a_{ij})$ be an $n \times n$ matrix. Derive the complete expansion of determinant of A. 7

3. a) Prove that the following conditions are equivalent for a square matrix A 8
   1. A can be reduced to the identity matrix by applying elementary operations
   2. A is a product of elementary matrices
   3. A is invertible
   4. The system $AX = 0$ has only the trivial solution.

   b) Prove that for any $n \times n$ matrix A, $A(\text{adj}A) = \det(A)I_n$. 6

4. a) If A is an $n \times n$ matrix with entries in a field F, show that the columns of A forms a basis of F^n if and only if A is invertible. 6

   b) If S and L are finite subsets of a vector space V such that S spans $V$and L is linearly independent, then show that S contains atleast as many elements as L does. Deduce that any two bases of a finite dimensional vector space V has the same number of elements; 8

5. a) Let V be an $n-$dimensional vector space and let $\mathcal{B}$ be an ordered basis of V. Prove that the collection of all ordered bases of V is $\{\mathcal{B}P: P \in GL_n(F)\}$. 8

   b) Define the matrix of change of basis. Prove that the matrix of change of basis is invertible. Determine the matrix of change of basis when the old basis is $(e_1, e_2, \ldots, e_n)$ and the new basis is $(e_n, e_{n-1}, \ldots, e_2, e_1)$ for $\mathbb{R}^n$. 6

6. a) If W_1 and W_2 are subspaces of a finite dimensional vector space Vover a field F then prove that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ 8

   b) If $W$is a subspace of a finite dimensional vector space$V$ over a field F then prove that there exists a subspace $W'$of$V$such that $V = W \oplus W'$. 6

7. a) Show that a map $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation about the origin in $\mathbb{R}^2$ if and only if the matrix of ρ with respect to the standard basis is in $SO_2(\mathbb{R})$. 6

   b) For a linear operator T on a finite dimensional vector space V, prove that $\dim(\ker T) + \dim(\text{im}\, T) = \dim V$. Hence show that a linear operator on a finite dimensional vector space is one-one if and only if it is onto. 8

Contd.. 2

8. a) Let $T : V \longrightarrow W$ be a linear map of the vector spaces V, W over a field F of dimensions n, m respectively. Prove that there exist bases $\mathcal{B}, \mathcal{C}$ of V, W respectively, such that the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{m\times n}$, where $r = \operatorname{rank} T$. 8

b) Compute the characteristic polynomial, eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. 6

*****

PH 563.1 **Reg No :** ☐☐☐☐☐☐☐

ST ALOYSIUS COLLEGE (AUTONOMOUS) MANGALURU

Semester I - P.G. Examination - M.Sc. Mathematics

November/December - 2023

REAL ANALYSIS I

Time : 3 Hours **Max. Marks : 70**

Answer FIVE FULL questions **(14x5=70)**

1. a) Let $A = \{p \in \mathbb{Q} : p > 0, p^2 < 2\}$ and $B = \{p \in \mathbb{Q} : p > 0, p^2 > 2\}$. Show that B has no greatest lower bound in $\mathbb{Q}$. 5

   b) If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_n \supseteq I_{n+1}, n = 1, 2, \ldots$ then prove that $\bigcap_{n=1}^{\infty} I_n$ is non-empty. 5

   c) Show that a finite subset of a metric space X has no limit points in X. 4

2. a) Prove that every $k-$cell in $\mathbb{R}^k$ is compact. 8

   b) Let X be a metric space and $Y \subseteq X$. Prove that a subset E of Y is open relative to Y if and only if $E = G \cap Y$ for some open set G of X. 6

3. a) Prove that compact subsets of a metric space are closed. 6

   b) If E is a subset of $\mathbb{R}^k$, then prove that the following properties are equivalent 8
   i). E is closed and bounded
   i). E is compact.

4. a) State and prove the Ratio test. Also test the convergence of the series $x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ 6

   b) Prove that $\lim_{n\to\infty}(1 + \frac{1}{n})^n = e$. Also prove that e is irrational. 8

5. a) State and prove the comparison test. If $|x| < 1$, then prove that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ 8

   b) Define a subsequence. If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X. 6

6. a) State and prove the comparison test. If $|x| < 1$, then prove that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ 6

   b) (i) Suppose X, Y, Z are metric spaces, $E \subseteq X$, f maps from E into Y, g maps the range of f and h is the mapping of E into Z defined by $h(x) = g(f(x)), \forall x \in E$. If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then show that h is continuous at p. 8

   (ii) Show that continuous image of a connected set is connected.

7. a) If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then prove that $f(X)$ is closed and bounded. Also show that f is bounded. 8

   b) Prove that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y. 6

8. a) State and prove chain rule for differentiation. 6

   b) State and prove Taylor's theorem. 8

*****

PS 564.1 Reg. No: | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester I – P.G. Examination - M. Sc. Mathematics
November/December - 2023

GRAPH THEORY

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Prove that in a graph G, closed walk of odd length contains an odd cycle
   b) Define a bipartite graph. Prove that graph G is bipartite if and only if all cycles are even **(6+8)**

2. a) Define the complement $\overline{G}$ of a graph G. For any graph G, with 6 points, prove that G or $\overline{G}$ contains a triangle.
   b) Show that the maximum number of lines among all p point graphs with no triangle is $\left[\frac{p^2}{4}\right]$, where p is even. **(4+10)**

3. a) Prove that a cubic graph has a cutpoint if and only if it has a bridge.
   b) Let G be a connected graph with at least 3 points. If G is a block then prove that every point and line of G lie on a common cycle. **(8+6)**

4. a) For a (p, q) graph G, prove that the following are equivalent:
   i) G is a tree
   ii) Every two points of G are joined by a unique path
   iii) G is connected and $p = q + 1$
   iv) G is acyclic and $p = q + 1$
   b) Prove that every tree has a center consisting of either one point or two adjacent points. **(8+6)**

5. State and prove Menger's theorem. **(14)**

6. a) Define a planar graph. Is K_5 and $K_{3,3}$ planar? Justify. If G is a plane map with p vertices, q edges and r faces then prove that $p - q + r = 2$.
   b) Prove that any planar graph is 5 colorable. **(6+8)**

7. a) Prove that every planar graph G with $p \geq 4$ points has atleast four points of degree not exceeding 5.

Contd...2

b) Prove that the following are equivalent for a connected graph G :

i) G is Eulerian

ii) every point of G is of even degree

iii) the set of lines of G can be partitioned into cycles. **(6+8)**

8 a) If G is a (p,q) plane graph with k components then prove that $p - q + r = k + 1$. Define maximal planar graph. If G is a (p,q) maximal plane graph then prove that every face is a triangle and $q = 3p - 6$.

b) Prove that, for every graph G with $'p'$ points, $\frac{p}{\beta_0} \le \chi(G) \le p - \beta_0 + 1$ where β_0 is the point independence number of G. **(7+7)**

****

PS 567.1 Reg No : ☐☐☐☐☐☐☐

ST ALOYSIUS COLLEGE (AUTONOMOUS) MANGALURU

Semester I - P.G. Examination - M.Sc. Mathematics

November/December - 2023

ORDINARY DIFFERENTIAL EQUATIONS

Time : 3 Hours | Max. Marks : 70

Answer FIVE FULL questions | (14x5=70)

1. a) Describe the method of variation of parameters for finding a particular solution of non- homogeneous linear second order equation. 8

   b) Using the method of reduction of order obtain the general solution of $x'' - 2tx' + 2x = 0$. 6

2. a) Prove that n solutions $\phi_1, \phi_2, \ldots, \phi_n$ of $x^n + a_1(t)x^{n-1} + \ldots + a_n(t)x = 0$ on I are linearly independent over I if and only if $W(\phi_1, \phi_2, \ldots, \phi_n)(t) \neq 0$ for all $t \in I$. 8

   b) Find the general solution of $t^2x' - 2tx' + 2x = t^3 \sin t$. 6

3. a) State and prove Abel's formula for n^{th} order linear homogeneous differential equation. 6

   b) Solve $x^{(4)} + 4x = 2sint + 1 + 3t^2 + 4e^t$. 8

4. a) Define the Legendre polynomial of degree n and show it is given by the Rodrigues formula $P_n(x) = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2 - 1)^n$. 8

   b) Find the Legendre series expansion of e^x. 6

5. a) State and prove the orthogonal property of the Legendre polynomials. 6

   b) Obtain $J_p(t)$ as the solution of the Bessel's equation $t^2x'' + tx' + (t^2 - p^2)x = 0$ where p is not an integer. 8

6. a) Show that $\Phi(t) = \begin{pmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{pmatrix}$ is a fundamental matrix of the system $X' = A(t)X$ where, $A(t) = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}$. 5

   b) Prove that the set of all solutions of the system $X' = A(t)X,\ t \in I$ forms an $n-$ dimensional vector space over the field of complex numbers. 4

   c) Let Φ be a fundamental matrix of the system $X'(t) = A(t)X(t)$ and $\psi(t) = \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds$, particular solution of $X'(t) = A(t)X(t) + b(t), t \in I$ with $x(t_0) = 0$. If $x_h(t)$ is a solution of the initial value problem $X'(t) = A(t)X(t), x(t_0) = x_0, t_0, t \in I$ then prove that $F(t) = x_h(t) + \psi(t)$ is also a solution $X'(t) = A(t)X(t) + b(t), x(t_0) = x_0$. 5

7. a) Let Φ be the fundamental matrix of $X'(t) = A(t)X(t)$ and Let C be a constant nonsingular matrix then prove that ΦC is also a fundamental matrix . Also prove that any fundamental matrix of the given system is of this type for some constant non-singular matrix C 5

   b) Let $A(t)$ be an $n \times n$ continuous matrix on I and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $x' = A(t)x$, then show that $\Phi(t + \omega)$ is also a fundamental matrix. 4

c) If A is an $n \times n$ constant matrix, prove that e^{tA} is a fundamental matrix of the system $x' = Ax$. 5

8. a) State and prove Picard's theorem 8

b) Let $f(t, x)$ be a continuous function defined over a rectangle $R = \{(t, x) : |t - t_0| \leq p, |x - x_0| \leq q\}$ where p, q are some positive real numbers. Let $\frac{\partial f}{\partial x}$ be defined and continuous on R then prove that $f(t, x)$ satisfies Lipschitz condition on R.is converse true justify? 6

*****