PH 561.4 Reg No :

St Aloysius College (Autonomous) Mangaluru

SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

MEASURE THEORY AND INTEGRATION

Time : 3 Hours **Max. Marks : 70**

Answer FIVE FULL questions **(14x5=70)**

1. a. Prove that the following statements regarding a subset E of $\mathbb{R}$ are equivalent: 6

   1. E is measurable.
   2. For every $\epsilon > 0$, there exists an open set O containing E such that $m^*(O-E) < \epsilon$.
   3. There exists a G_δ set, $G \supseteq E$ such that $m^*(G-E) = 0$.

   b. Define a measurable function. Prove that the following statements are equivalent: 8

   1. f is a measurable function.
   2. For each $\alpha \in \mathbb{R}, \{x : f(x) \geq \alpha\}$ is measurable.
   3. For each $\alpha \in \mathbb{R}, \{x : f(x) < \alpha\}$ is measurable.
   4. For each $\alpha \in \mathbb{R}, \{x : f(x) \leq \alpha\}$ is measurable.

2. a. If f, g are real valued measurable functions defined on the same measurable set E and c is any real number, then prove that $f+c$, fc and $f+g$ are measurable. 5

   b. If f is a measurable function and $f = g \; a.e.$ then prove that g is measurable. 4

   c. If $\{f_n\}$ is a sequence of measuable functions defined on the same measurable set then show that $\sup f_n$ and $\inf f_n$ are measurable. 5

3. a. Show that every finite set has zero outer measure. 4

   b. If f and g are measurable functions then prove the following: 5

   1. $\text{ess}\sup(f+g) \leq \text{ess}\sup f + \text{ess}\sup g$.
   2. $\text{ess}\sup f = -\text{ess}\inf(-f)$.

   c. Show that for any subset A of $\mathbb{R}$, $m^*(A) = m^*(A+x)$ where $A+x = \{a+x \mid a \in A\}$. 5

4. a. If A and B are disjoint measurable sets and f is an integrable function then show that $\int_A f\,dx + \int_B f\,dx = \int_{A\cup B} f\,dx$. 5

   b. Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|\,dx < +\infty$. Then prove that the series $\sum_{n=1}^{\infty} f_n$ converges $a.e.$, its sum f is integrable and $\int f\,dx = \sum_{n=1}^{\infty} \int f_n\,dx$. 4

   c. State Fatou's lemma. State and prove the Lebesgue monotone convergence theorem. 5

Contd....2

5. a. Define integral of a non-negative measurable function. Show that if f is a non-negative measurable function then $f = 0\ a.e.$ if and only if $\int f\,dx = 0$. 8

b. Show that if f and g are measurable, $|f| \leq |g|\ a.e.$ and g is integrable then f is integrable. 6

6. a. Show that a convex function on any interval (a, b) is continuous. 6

b. State and prove Jensen's inequality. 8

7. a. State and prove Minkowski's inequality. 5

b. Show that $L^p(X, \mu)$ is a vector space over $\mathbb{R}$. 4

c. If ψ is convex on (a, b) and $a < s < t < u < b$ then show that $\psi(s,t) \leq \psi(s,u) \leq \psi(t,u)$. 5

8. a. State and prove the Hahn decomposition theorem. 8

b. Define a signed measure. If $[|X, \mathcal{S}, \mu|]$ is a measure space and f is a non-negative measurable function then prove that $\phi(E) = \int_E f\,d\mu$ is a measure on the measurable space $[|X, \mathcal{S}|]$. 6

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SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Complex Analysis II

Max. Marks : 70

Time : 3 Hours

(14x5=70)

Answer FIVE FULL questions

1. a. i) Let Ω be a multiply connected region of connectivity n then show that every cycle γ in Ω is homologous to a linear combination of the cycles $\gamma_1, \gamma_2, \ldots, \gamma_{n-1}$ and the linear combination is uniquely determined. 6

ii) Illustrate an annulus and show that the integral of an analytic function over a cycle is a multiple of a single period whose value is independent of the radius.

b. Prove that a region is simply connected if and only if $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω. 8

2. a. State and prove the Residue theorem. 8

b. Find the residue of the function $f(z) = \frac{e^z}{(z-a)(z-b)}$. Also show that the residue at a removable singularity is zero. 6

3. a. Evaluate 5

i) $\displaystyle\int_0^{2\pi} \frac{1}{2 - cos\theta} d\theta$

ii) $\displaystyle\int_c \frac{e^z}{z^4 + 5z^3} dz, \ \ c : |z| = 2$

b. Compute $\displaystyle\int_0^{\pi} \frac{d\theta}{a + cos\theta}, \ \ a > 1$ 5

c. Evaluate $\displaystyle\int_c \frac{cosz}{(z - \pi i)^2} dz, \ \ c : |z| = 5$ 4

4. a. Prove that a non-constant harmonic function has neither a maximum nor a minimum in its region of definition. Also prove that the maximum and minimum on a closed bounded set E are taken on the boundary of E. 8

b. Define a piecewise continuous function and Poisson's integral of a piecewise continuous real valued function u defined on a unit circle. Suppose $f(z)$ is analytic in the whole plane, real on the real axis and purely imaginary on the imaginary axis, then show that $f(z)$ is an odd function. 6

5. a. If u_1 and u_2 are harmonic in a region Ω, then prove that $\int_\gamma u_1^* du_2 - u_2^* du_1 = 0$ for every cycle γ which is homologous to zero in Ω. 6

b. State and prove Poisson's formula. 8

6. a. If the functions $f_n(z)$ are analytic and non-zero in a region Ω, and if $f_n(z)$ converges to $f(z)$ uniformly on every compact suset of Ω, then prove that $f(z)$ is either identically zero or never equal to zero in Ω. 8

b. Show that $\lim\limits_{n\to\infty} (1 + \frac{z}{n})^n = e^z$ uniformly on every compact subset of the complex plane. 6

7. a. If $f(z)$ is analytic in a region Ω containing z_0, then show that the representation $f(z) = f(z_0) + f'(z_0)\frac{z-z_0}{1!} + \ldots + f^n(z_0)\frac{(z-z_0)^n}{n!} + \ldots$ is valid in the largest open disk of center z_0 contained in Ω. 8

b. State and prove Weirstrass theorem. 6

8. a. Show that $\frac{\pi^2}{sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$. Deduce that $\pi cotz = \frac{1}{z} + \sum_{n\neq 0}[\frac{1}{z-n} + \frac{1}{n}]$. 8

b. State and prove Mittag-Leffler's theorem. 6

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SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Functional Analysis

Time : 3 Hours Max. Marks : 70

Answer **FIVE FULL** questions (14x5=70)

1. a. State and prove the Baire's category theorem. 8

   b. State and prove the Cantor's intersection theorem. 6

2. a. State and prove Holder's inequality and hence deduce Minkowski's inequality for $n-$tuples of scalars. 6

   b. Define the equivalence of two norms in a linear space. Let L be a linear space made into a normed linear space by $||.||$ and $||.||'$. Show that these two norms are equivalent if and only if there exist positive reals K_1 and K_2 such that $K_1||x|| \leq ||x||' \leq K_2||x||$, for all $x \in L$. 8

3. a. Let M be a closed linear subspace of a normed linear space N. Prove that N/M forms a normed linear space with respect to the norm given by $||x+M|| = \inf\{\ ||x+m|| : m \in M\}$, for every $x+M \in N/M$. 8

   b. Prove that the set of all bounded linear transformations $\mathcal{B}(N, N')$ of a normed linear space N into a normed linear space N' forms a normed linear space with respect to the norm given by $\|T\| = \sup\{\|T(x)\| : x \in N, \|x\| \leq 1\}$. 6

4. a. Define the conjugate space N^* of a normed linear space N. Show that N can be embedded in N^{**}. 6

   b. Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ is a nonempty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector $x \in B$, then prove that $\{\|T_i\|\}$ is a bounded set of numbers. 8

5. State and prove the open mapping theorem. 14

6. a. If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$. 5

   b. If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace $M + N$ is also closed. 5

   c. State and prove the parallelogram law in a Hilbert space H. Does the Parallelogram law hold in the Banach space l_1^n? justify your answer. 4

7. a. Show that a unitary operator T is an isometric isomorphism of H into itself. Is the converse true? Justify. 8

   b. Define an orthonormal set in a Hilbert space H. For a finite orthonormal set $\{e_1, e_2, \ldots, e_n\}$ in a Hilbert space H and $x \in H$, show that $x - \sum_{i=1}^{n} \langle x, e_i \rangle \perp e_j$, for each $j, 1 \leq j \leq n$. 6

8. a. Define the orthogonal complement $M^\perp$ of a subspace M of a Hilbert space H. Show that $M^\perp$ is a closed linear subspace of H. If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^\perp$. 8

   b. Let H be a Hilbert space. If T is an operator on a Hilbert space H for which $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T = 0$. Further prove that an operator T on a Hilbert space is self-adjoint if and only if $\langle Tx, x \rangle$ is real for all $x \in H$. 6

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PS 565.4

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SEMESTER IV - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Partial Differential Equations

Time : 3 Hours — Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. Prove that a necessary and suffiecient condition for a differential equation $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = 0$ to be integrable is that $X.CurlX = 0$. 8

   b. Test for integrability of $(y+z)\,dx + (x+z)\,dy + (x+y)\,dz = 0$ and find its primitive. 6

2. a. Prove necessary and suffcient condition that there exists between two functions $U(x,y)$ and $V(x,y)$ a relation $F(U,V) = 0$ not involving x or y explicitly is $\frac{\partial(U,V)}{\partial(x,y)} = 0$. 6

   b. Test for integrability of $yz(1+4xz)\,dx - xz(1+2xz)\,dy - xy\,dz = 0$ and find its primitive. 8

3. a. Find the characteristic of the equation $p^2 + q^2 = 2z$ and determine the integral surface which passess through the circle $x^2 + y^2 = 1$ and $u = 1$. 8

   b. Derive a necessary condition for the compatibility of $f(x,y,u,p,q) = 0$ and $g(x,y,u,p,q) = 0$. 6

4. a. Obtain the partial differential equation by eliminating the arbitrary function f from the following, 6
   i) $f(x^2 + y^2 + z^2, z^2 - 2xy) = 0$.
   ii) $u = f(\frac{xy}{z})$.

   b. Find the integral surface of the linear partial differential equation $2y(u-3)u_x + (2x-u)u_y = y(2x-3)$ which passes through the circle $x^2 + y^2 = 2x, u = 0$. 8

5. a. Find the family of surfaces which is orthogonal to one parameter family of surfaces $z(x+y) = c(3z+1)$, where c is a constant, which passes through the circle $x^2 + y^2 = 1, z = 1$. 8

   b. Solve $(D^2 - DD')u = cosxcos2y$. 6

6. a. Solve $(D^2 + DD' - 6D'^2)u = ycosx$. 8

   b. Solve $(3D^2 - D')u = sin(x+y)e^x$. 6

7. a. Obtain the solution of the wave equation $u_{tt} = c^2 u_{xx}$ under the following conditions 6
   i) $u(0,t) = u(2,t) = 0$ ii) $u(x,0) = sin(\frac{\pi x}{2})$ iii) $u_t(x,0) = 0$

   b. Classify the equation $u_{xx} - 2sinxu_{xy} - cos^2xu_{yy} - cosxu_y = 0$ and reduce it into cannonical form. 8

8. a. Solve the one dimentional diffusion equation $T_t = c^2 T_{xx}$ in the region $0 \leq x \leq \pi,\ t \geq 0$ subject to 8

i) T remains finite as $t \to \infty$

ii) $T = 0$ if $x = 0$ and $x = \pi, \forall t$

iii) At $t = 0,\ T = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$

b. Obtain the D'Alemberts solution of the initial value problem of Cauchy type described as $z_{tt} - C^2 z_{xx} = 0, \quad -\infty < x < \infty,\ t > 0$, initial conditions $z(x,0) = f(x), z_t(x,0) = g(x)$, where f and g are twice continuously differentiable functions on $\mathbb{R}$. 6

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