PH 561.2

Reg No :

St Aloysius College (Autonomous) Mangaluru

SEMESTER II - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Algebra II

Time : 3 Hours

Max. Marks : 70

(14x5=70)

Answer FIVE FULL questions

1. a. Prove that in an integral domain existence of factorisation holds if and only if every increasing chain of principal ideals is stationary. 8

   b. Prove that every Euclidean domain is a principal ideal domain. 6

2. a. Define primitive polynomial. Prove that every nonzero polynomial $f(x) \in \mathbb{Q}[x]$ can be writen as a product of a rational number and a primitive polynomial in $\mathbb{Z}[x]$ 6

   b. State and prove Eisenstein's criterion for irreducibility of a polynomials in $\mathbb{Q}[x]$. Verify whether the polynomial $x^5 - 64x^4 + 127x^3 - 200x + 99$ is irreducible in $\mathbb{Q}[x]$. 8

3. a. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be an integer polynomial and p be a prime integer such that $p \nmid a_n$. If the residue $\bar{f}$ of f modulo p is an irreducible element in $F_p[x]$, then prove that f is irreducible element in $\mathbb{Q}[x]$. 6

   b. Prove that the polynomial ring $\mathbb{Z}[x]$ is a unique factorization domain. 8

4. a. If $f(x)$ is a polynomial over a field F prove that $f(x)$ has no multiple roots in any extension of F if and only if $f(x)$ and $f'(x)$ are relatively prime in $F[x]$. 6

   b. Let L and K be extensions of a field F . Let $\alpha \in L$ and $\beta \in K$ be algebraic over F. Prove that there exists an F-isomorphism from $F(\alpha)$ to $F(\beta)$ if and only if α and β are the roots of the same irreducible polynomial over F. 8

5. a. If $F \subseteq K \subseteq L$ are field, then prove that $[L : F] = [L : K][K : F]$. Also Determine $[\mathbb{Q}(3\sqrt{2}, 4\sqrt{5}) : \mathbb{Q}]$ 8

   b. Let K be an extension of field F. Prove that, the elements of K which are algebraic over F forms a subfield of K containing F. 6

6. a. Let F be a field with p^n elements and m be a positive integer. Prove that F has a subfield with p^m elements if and only if $m|n$. Also determine the number of subfields of a field with 2^{2024} elements. 8

   b. Define finite field and prove that characteristic of a finite field is a prime. 6

7. a. Define algebraic closure and prove that if F is a finite field, then F cannot be algebraically closed. 6

   b. Show that the set of all constitutible real numbers form a subfield of $\mathbb{R}$ containing $\mathbb{Q}$. 8

8. a. State and prove the fundamental theorem of Galois theory. 8

   b. If K is a Galois extension of a field F , then prove that the fixed field of K is F . 6

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SEMESTER II - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Real Analysis II

Time : 3 Hours Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $a < c < b$, then prove that $f \in \mathscr{R}(\alpha)$ on $[a, c]$ and $[c, b]$. Also prove that $\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha$. 8

   b. Suppose f is bounded on $[a, b]$, f has finitely many points of discontinuity on $[a, b]$ and α is continuous at every point at which f is continuous. Then prove that $f \in \mathscr{R}(\alpha)$. 6

2. a. Let f be a bounded function and let α be a monotonically increasing function on $[a, b]$, then show that $\underline{\int_a^b} f\,d\alpha \leq \overline{\int_a^b} f\,d\alpha$. Also if $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M, \forall$ $x \in [a, b]$, then prove that $|\int_a^b f\,d\alpha| \leq M(\alpha(b) - \alpha(a))$. 6

   b. i) Suppose $f \in \mathscr{R}$ on $[a, b]$ where $f: [a, b] \rightarrow \mathrm{R}^k$. Define $F(x) = \int_a^x f(t)\,dt, a \leq x \leq b$. Prove that $F: [a, b] \rightarrow \mathrm{R}^k$ is continuous. Also if f is continuous at x_0, then prove that F is differentiable at x_0 and $F'(x_0) = f(x_0)$. 8

   ii) State and prove the Fundamental theorem of Vector Calculus.

3. a. Define uniform metric. Prove that $C(X)$ is a complete metric space with the metric defined by $d(f, g) = ||f - g||$, $\forall f, g \in C(X)$. 6

   b. Define a equicontinuous family of complex valued functions. If K is a compact metric space, if $f_n \in C(K), n = 1, 2, \ldots$ and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equicontinuous on K. 8

4. a. Define a pointwise bounded sequence. If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then prove that $\{f_n\}$has a convergent subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$. 6

   b. Prove that there exists a real continuous function on the real line which is nowhere differentiable. 8

5. a. Let B be the uniform closure of algebra A of bounded functions on a set E. Then prove that B is a uniformly closed algebra. 6

   b. State and prove the uniform convergence and continuity theorem. Is the converse true? Justify. 8

6. a. State and prove Cauchy's test. Furthermore if a function ϕ is bounded in $[a, \infty)$ and integrable in $[a, x], x \geq a$ and if $\int_a^\infty f\,dx$ is absolutely convergent then prove that $\int_a^\infty f\phi\,dx$ is also absolutely convergent. 4

   b. If f and g are two functions on $[a, b]$ such that $lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = l$ where l is a non-zero finite number then prove that the two integrals $\int_a^b f\,dx$ and $\int_a^b g\,dx$ converge or diverge together at a. 5

   c. Examine the convergence of $\int_0^2 \frac{dx}{2x - x^2}$ 5

Contd...2

7. a. If $\phi(x)$ is bounded and monotonic in $[a, \infty)$ and tends to zero as $x \to \infty$ and $\int_a^x f\,dx$ is bounded for $x \geq a$ then prove that $\int_a^\infty f\phi\,dx$ is convergent. 4

b. Check the convergence of: 5

i) $\int_0^\pi \frac{sinx}{x^3} dx$

ii) $\int_0^1 logx dx$

c. Prove that every absolutely convergent integral is convergent. 5

8. a. Suppose f is a continuous-differentiable mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$, then prove that there exist open sets U, V in R^n such that $a \in U, b \in V$, f is one-one on U and $f(U) = V$. 8

b. Suppose f is a continuous-differentiable mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. If g is the inverse of f defined in V by $g(f(x)) = x$, then prove that $g \in C'(V)$. 6

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St Aloysius College (Autonomous) Mangaluru

SEMESTER II - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Research Methodology and Ethics

Time : 3 Hours **Max. Marks : 70**

Answer FIVE FULL questions **(14x5=70)**

1. a. Discuss the importance of defining research objectives and hypotheses in the research process. 6

   b. Write a short note on: 8
      i) Objectives of research
      ii) motivation in research

2. a. What is the purpose of research, and why is it important in advancing knowledge in various fields? 8

   b. Explain the benefits of literature review in research. 6

3. a. Describe the techniques involved in defining a research problem. 6

   b. Describe various types of research with example. 8

4. a. Define and explain the concept of a mathematical definition. Discuss the importance of precise definitions in mathematics and their role in establishing the framework for mathematical reasoning. 6

   b. What is Latex? Explain its role in mathematical documentation. 8

5. a. Explain the essential rules involved in preparing a mathematical document. 6

   b. Choose a significant mathematical theorem or result related to your field of study or interest and explain its statement and significance. 8

6. a. Describe the concept of Intellectual Property Rights. Explain its origin and importance. 6

   b. Write the meaning and importance of patents, copyrights, and trademarks. 8

7. a. What are the key principles of research ethics, and why are they important in scientific inquiry? 6

   b. What does scientific misconduct refer to? Discuss the various forms of scientific misconduct. 8

8. a. Explain the concept of copyright and its importance in academic and creative works. What are various subjects covered under copyrights? explain. 8

   b. Describe the roles of co-authors, contributors, and acknowledgments in scholarly publications. Provide examples of ethical dilemmas related to authorship and contributorship. 6

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PS 564.2 Reg No : ☐☐☐☐☐☐☐

St Aloysius College (Autonomous) Mangaluru

SEMESTER II - PG EXAMINATION - M.Sc Mathematics

MAY - 2024

Linear Algebra II

Time : 3 Hours Max. Marks : 70

Answer FIVE FULL questions (14x5=70)

1. a. Let $(w_1, w_2, \ldots\ldots, w_k)$ be an orthonormal basis of a subspace W of V and let $v \in V$. Then show that the orthogonal projection $\pi(v)$ is a vector given by $\pi(v) = < v, w_1 > w_1 + < v, w_2 > w_2 + \ldots\ldots + < v, w_k > w_k$. 5

b. Define the following terms : 4
   1. Hermitian form on a finite dimensional complex vector space V.
   2. Positive definite Hermitian form.

c. If A is a non-singular skew-symmetric $m \times m$ matrix then prove that m is even and there exists $Q \in GL_m(F)$ such that $QAQ^t = J_{2n} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ where $n = \frac{1}{2}m$. 5

2. a. Find an orthogonal basis for the form X^tAY on $\mathbb{R}^n$ where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. 6

b. If A is a $n \times n$ real symmetric matrix then show that e^A is symmetric and positive definite. 8

3. State and prove Sylvester's law. 14

4. a. If T is a linear operator on a Hermitian space V and T^* is the adjoint operator then prove that T is normal if and only if $< T(v), T(w) > = < T^*(v), T^*(w) >, \forall v, w \in V$. 5

b. Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form X^tAY where $A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$. 5

c. Define orthogonal complement $W^\perp$ of a subspace W of a vector space V. Show that $W^\perp$ is a subspace of V. 4

5. a. If M is a $R-$module and N is a submodule of M then prove that $\eta: M \to M/_N$ defined by $\eta(x) = x + N, \forall x \in M$, is a surjective module homomorphism with $\ker \eta = N$. 4

b. If N is a submodule of M then prove that there exists a bijective correspondence between the set of all submodules of M which contains N and the set of all submodules of $M/_N$. 5

c. If M is a R-module and N is a submodule of M such that both N and M are finitely generated, then show that the number of elements in a generating set for N need not be less than or equal to the the number of elements in a generating set for M. 5

6. a. If V is a simple R-module then prove that the set of all endomorphisms of V is a field. 6

b. Define a free R-module. If M is a finitely generated R-module then prove that M is isomorphic to a quotient of R^n for some $n \in \mathbb{N}$. 8

Contd...2

7.

a. Find the integer solutions of $AX = 0$ where $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & -3 & 1 \\ -4 & 6 & -2 \end{bmatrix}$. 8

b. Define a Noetherian module. If R is a Noetherian ring then show that every proper ideal of R is contained in a maximal ideal of R. 6

8. a. If M is an R-module and N is a submodule of M then prove that $M/N = \{x + N : x \in M\}$ is a R-module. Also, state and prove the first isomorphism theorem for modules. 8

b. Determine the abelian group presented by the matrix $A = \begin{bmatrix} 3 & 8 & 7 & 9 \\ 2 & 4 & 6 & 6 \\ 1 & 2 & 2 & 1 \end{bmatrix}$. 6

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