PH 561.1 Reg. No. [| | | | | |]

St. Aloysius College (Autonomous)

Semester I – P.G. Examination – M. Sc. Mathematics

February 2021

ALGEBRA - I

Time : 3 Hours Max. Marks : 70

Answer any FIVE FULL questions from the following. $(14 \times 5 = 70)$

1. (a) State and prove the Lagrange's theorem for finite groups.

   (b) Let G be a cyclic group of order n and $a \in G$ be a generator of G. Prove that a^k (k is an integer, $1 \leq k \leq n$) is a generator of G if and only if $\gcd(k, n) = 1$. **(8+6)**

2. (a) If H and K are normal subgroups of a group G with $G = HK$ and $H \cap K = \{e\}$, where e is the identity element in G, then show that G is isomorphic to $H \times K$.

   (b) Prove or disprove the following:

   (i) Subgroup of a cyclic group is cyclic.

   (ii) A group with no proper non-trivial subgroup is cyclic.

   (c) If H is a subgroup of index 2 in a group G, then show that H is a normal subgroup of G.

   (6+6+2)

3. (a) Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be any function. Then prove that the following are equivalent.

   i. ϕ is an isometry which fixes the origin.

   ii. ϕ preserves the dot product

   iii. ϕ is an orthogonal linear operator

   (b) Show that orientation-preserving isometries of a plane are translations and rotations

   (7+7)

4. (a) Define a group action(operation) on a set. Derive the counting formula for a group G acting on a set S.

   (b) Prove that the group $GL_2(\mathbb{F}_2)$ of invertible matrices with mod 2 coefficients is isomorphic to the symmetric group S_3.

   (c) Let H and K be subgroups of a group G. Prove that the index of $H \cap K$ in H is at most equal to the index of K in G. (5+5+4)

5. (a) State and prove the third Sylow theorem for finite groups.

Contd... 2

(b) Show that every group of order 15 is cyclic. (8+6)

6. (a) Determine the class equation for the symmetric group S_4.

(b) Prove that the center of a p-group G has order > 1.

(c) Prove that every group of order p^2 is abelian. (5+5+4)

7. (a) Let R be a commutative ring with unity $1 \neq 0$. Prove that an ideal M in R is a maximal ideal if and only if R/M is a field.

(b) Prove that every finite integral domain is a field.

(c) Define a prime ideal in a ring. Determine all the prime ideals of the ring $\mathbb{Z}$. (6+4+4)

8. (a) Prove that every integral domain can be embedded in a field.

(b) Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$. (10+4)

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PH 562.1

Reg. No. | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester I - P.G. Examinations - M.Sc. Mathematics
February 2021

LINEAR ALGEBRA-I

Time : 3 Hours Max. Marks : 70

Answer any FIVE FULL questions from the following: $(14 \times 5 = 70)$

1. (a) Find a formula for A^n and prove it by induction, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) Prove that a square row echelon matrix is either the identity matrix or else its bottom row is zero. Hence derive that a square matrix is invertible if and only if its row echelon form is the identity matrix.

   (c) Let A be a square matrix that has a left inverse B. Show that A is invertible and $A^{-1} = B$. $(5+5+4)$

2. (a) For a square matrix A, prove that the following are equivalent:

   i. A can be reduced to the identity matrix by appying elementary row operations
   ii. A is aproduct of elememntary matrices
   iii. A is invertible
   iv. the system $AX = 0$ has only the trivial solution

   (b) Prove that $det(AB) = det(A)det(B)$, for any two $n \times n$ matrices A and B.

   (c) Is $det(A + B) = det(A) + det(B)$, for any two $n \times n$ matrices A and B?.Justify

   $(6+6+2)$

3. (a) Prove that $\det A = \det A^t$ for any square matrix A.

   (b) Prove that $\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} = (\det A)(\det D)$ if A and D are square blocks.

   (c) Prove that a square matrix P is a permutation matrix if and only if it has the single non-zero entry 1 in each row and in each column. $(5+5+4)$

4. (a) Let S and L be finite subsets of a vector space V. Assume that S spans V and that L is linearly independent. Show that $|L| \leq |S|$.

   (b) Let V be a finite dimensional vector space over a field F. If W is any subspace of V, prove that W is also finite dimensional and $\dim_F W \leq \dim_F V$. $(7+7)$

5. (a) Let W be a subspace of a vector space V over a filed F.

   (i) Prove that there is a subspace U of V such that $U + W = V$ and $U \cap W = \{0\}$.

(ii) Prove that there is no subspace U such that $W \cap U = \{0\}$ and that $\dim W + \dim U > \dim V$.

(b) Determine the base change matrix in $\mathbb{R}^n$, when the old basis is the standard basis and the new basis is $\mathcal{B} = (e_n, e_{n-1}, \ldots, e_1)$.

(c) Let W_1 and W_2 be any two subspaces of a finite-dimensional vector space V. Prove that $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$. $(5+2+7)$

6. (a) Determine the rank of the matrix $\begin{bmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{bmatrix}$.

(b) Let A be the matrix of a linear transformation $T : V \to W$ with respect to some ordered bases $\mathcal{B}$ and $\mathcal{C}$ of V and W respectively. Prove that the matrices A' which represent T with respect different ordered bases of V and W are given by $A' = QAP^{-1}$, where $Q \in GL_m(F)$ and $P \in GL_n(F)$, with $m = \dim W$, $n = \dim n$.

(c) Prove that similar matrices have same eigenvalues. $(5+7+2)$

7. (a) Let $T : V \to V$ be a linear operator on a finite dimensional vector space V over a field F. Prove that the following are equivalent:

(i) $\ker T > 0$.

(ii) $\text{im}T < V$.

(iii) If A is the matrix of the operator with respect to an arbitrary basis, then $\det A = 0$.

(iv) 0 is an eigenvalue of T.

(b) Prove that the matrices $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ and $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are similar if and only if $a \neq d$.

(c) Do A and A^t have the same eigenvalue? the same eigenvectors? Justify. $(5+4+5)$

8. (a) Let A be a matrix in O_3 with $\det A = -1$. Prove that -1 is an eigenvalue of A.

(b) Let T be a linear operator on a vector space of dimension n over a field F, whose characteristic polynomial has n distinct roots in F.Prove that there is an ordered basis $\mathcal{B}$ of V with respect to which the matrix of T is diagonal.

(c) Compute e^A directly from the expansion, where $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. $(4+6+4)$

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PH 563.1

St Aloysius College (Autonomous)

Mangaluru

Semester I – P.G. Examination –M.Sc. Mathematics

February 2021

REAL ANALYSIS - I

Time : 3 hours **Max. Marks :70**

Answer any FIVE FULL questions from the following:

1. a) Define the notions of an upper bound and least upper bound of a subset E of an ordered set S.

   b) State and prove the Archimedean property of $\mathbb{R}$.

   c) State and prove Schwarz inequality.

   d) Let B be a non empty set of real numbers which is bounded above. Let $-B$ be the set of all numbers $-x$, where $x \in B$. Prove that $\sup B = -\inf(-B)$. **(2+4+4+4)**

2. a) Define a dense set. Prove that rational numbers are dense in $\mathbb{R}$.

   b) Define a countable set. Prove that the set $\mathbb{Z}$ of all integers is countable.

   c) Prove that the set of all sequences whose elements are the digits 0 and 1 is uncountable. **(5+5+4)**

3. a) If $x, y \epsilon \mathbb{R}$, prove that $d(x,y) = \frac{|x-y|}{1+|x-y|}$ is a metric on $\mathbb{R}$.

   b) Define a closed set in a metric space. Prove that a finite union of closed sets in a metric space is a closed set. Is an arbitrary union of closed sets closed? Justify your answer.

   c) Define a limit point in a metric space. In a metric space prove that a finite set has no limit points. **(4+5+5)**

4. a) Define a compact set. Prove that every $k-cell$ in $\mathbb{R}^k$ is compact.

   b) Define a connected set. Prove that a subset E of $\mathbb{R}$ is connected if and only if it is an interval. **(7+7)**

5. a) If $\{s_n\}$ and $\{t_n\}$ are complex sequences, such that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, prove that $\lim_{n\to\infty} s_n t_n = st$.

   b) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ for real $p > 0$ and real α.

Contd...2

c) Show that the set of all subsequential limits of a sequence $\{p_n\}$ in a metric space X forms a closed subset of X.

(3+7+4)

6. a) State and prove Ratio test for convergence of a series.

b) For any sequence $\{c_n\}$ of positive real numbers, prove that $\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$.

c) If $a_n > 0$ and $\sum a_n$ diverges, prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(6+4+4)

7. a) Let X and Y be metric spaces and $f: X \to Y$ be a function. Prove that f is continuous if and only if for every sequence $\{x_n\}$ converging to x in X, the sequence $\{f(x_n)\}$ converges to $f(x)$.

b) Define uniform continuity. Prove that a continuous mapping of a compact set X into a metric space Y is uniformly continuous.

c) Consider the function f defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} 0, & for \quad x \ irrational \\ \frac{1}{n}, & for\ x \neq 0\ rational\ x = \frac{m}{n}, n > 0, \quad gcd(m,n) = 1 \\ 1, & for \quad x = 0 \end{cases}$$

Prove that f is continuous at every irrational point and has a simple discontinuity at every rational point.

(4+6+4)

8. a) Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x \in [a,b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, $a \leq t \leq b$. Then prove that h is differentiable at x, and $h'(x) = g'(f(x))\, f'(x)$.

b) State and prove Taylor's theorem.

c) Suppose f is defined in a neighbourhood of x, and suppose $f''(x)$ exists. Show that $\lim_{h\to 0} \frac{f(x+h)-f(x-h)-2f(x)}{h^2} = f''(x)$.

(5+6+3)

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Semester I – P.G. Examination - M. Sc. Mathematics

February 2021

GRAPH THEORY

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Define the intersection number $\omega(G)$ of a graph G. Let G be a connected graph with $p > 3$ points. If G has no triangles, then show that $\omega(G)=q$, the number of lines of G.

   b) Define the complement $\overline{G}$ of a graph G. Prove or disprove: If G is connected, then $\overline{G}$ is disconnected.

   c) For any graph G, with 6 points, prove that G or $\overline{G}$ contains a triangle. **(7+3+4)**

2. Show that the maximum number of lines among all p point graphs with no triangle is $\left[\frac{p^2}{4}\right]$. **(14)**

3. a) Show that a cubic graph has a cutpoint if and only if it has a bridge.

   b) Let G be a connected graph with at least 3 points. If G is a block then prove that every point and line of G lie on a common cycle. **(6+8)**

4. a) For a (p, q) graph G, prove that the following are equivalent:

   i) G is a tree

   ii) Every two points of G are joined by a unique path

   iii) G is connected and $p = q + 1$

   iv) G is acyclic and $p = q + 1$

   b) Prove that, every non-trivial tree has at least two end points. **(11+3)**

5. State and prove Menger's theorem. **(14)**

6. a) Define an Eulerian graph and a Hamiltonian graph. Give an example of a graph which is Hamiltonian but not Eulerian.

   b) Let G be a graph having $p \geq 3$ points. If for every $n, 1 \leq n < \frac{p-1}{2}$, the number of points of degree not exceeding n is less than n and if, for odd p, the number of points of degree at most $\frac{p-1}{2}$ does not exceed $\frac{p-1}{2}$, then prove that G is Hamiltonian. **(3+11)**

7. a) If G is a plane map with p vertices, q edges and r faces then $p - q + r = 2$.

Contd...2

b) If G is any planar (p,q) graph with $p \geq 3$, having no triangle, then show that $q \leq 2p-4$.

c) Prove that the graphs K_5 and $K_{3,3}$ are non-planar. **(5+5+4)**

8 a) Prove that every planar graph is 5-colorable.

b) Prove that, for every graph G with 'p' points, $\frac{p}{\beta_0} \leq \chi(G) \leq p - \beta_0 + 1$ where β_0 is the point independence number of G. **(9+5)**

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OPERATIONS RESEARCH

Time: 3 hrs **Max Marks: 70**

Answer any FIVE full questions.

1. a) Suppose that 8, 12 and 19 units of proteins, carbohydrates and fats are the minimum weekly requirement for a person. Suppose food A contains 2, 6 and 1 units of proteins, carbohydrates and fats respectively per kg and food B contains 1, 1and 3 respectively per kg. If food A costs Rs. 85/kg and food B costs Rs. 40/kg, how many kilograms of each should be bought to minimize the cost and still meet the minimum requirements?

   b) Solve the given LPP by graphical method:
   Maximize $z = 3x_1 + 5x_2$
   Subject to
   $x_1 \le 4$
   $2x_2 \le 12$
   $3x_1 + 2x_2 \le 18$
   $3x_1 + 5x_2 \ge 50$
   $x_1, x_2 \ge 0$

   c) Write the standard form of the LPP
   Maximize $z = 3x_1 + 2x_2 + 5x_3$
   Subject to
   $2x_1 - 3x_2 \le 3$
   $x_1 + 2x_2 + 3x_3 \ge 5$
   $3x_1 + 2x_3 \le 2$
   $x_1, x_2 \ge 0$

(7+5+2)

2. A firm manufactures four different machines parts M_1, M_2, M_3 and M_4 made of copper and zinc. The amount of copper and zinc required for each machine part, their exact availability and profit earned from one unit of each machine part are given below

	M_1 (kg)	M_2 (kg)	M_1 (kg)	M_1 (kg)	Exact availability
copper	5	4	2	1	100
zinc	2	3	8	1	75
Profit (Rs.)	12	8	14	10	

Solve the given problem using analytical method. Also, find

(i) Basic solutions
(ii) Basic feasible solutions
(iii) Non degenerate basic feasible solutions
(iv) Optimal basic feasible solutions **(14)**

Contd...2

3. A company produces two types of leather belts, say A and B. But A is of superior quality and B is inferior. Profit on A and B is Rs. 4 and Rs. 3 respectively. Each belt of type A requires twice as much as time required by type B. If all the belts were of type B the company could produce 1000 belts per day. But the supply of leather is sufficient only for 800 belts per day. Belt A requires a fancy buckle and only 400 of them are available per day. For belt B, only 700 buckles are available per day. How many units of the two belts should be produced in order to have maximum profit? Solve by simplex method. **(14)**

4. a) Solve the given LPP

Maximize $z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to
$$x_1 + 2x_2 + x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

b) Write the dual of the LPP

Maximize $z = 3x_1 + x_2 + 2x_3 - x_4$

Subject to
$$2x_1 + x_2 + 3x_3 + x_4 = 1$$
$$x_1 + x_2 - x_3 + x_4 = 3$$
$$x_1, x_2 \geq 0$$

(12+2)

5. Give the transportation algorithm including the steps of the modified distribution method. **(14)**

6. A company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs differ from factory to factory and sales price from agency to agency. Also the shipping cost per unit product from each factory to each agency is known. Given the following data find the production and distribution schedule most profitable to the company.

Factory (i)	**Production cost /unit (Rs)**	**Maximum capacity (No. of units)**
1	18	140
2	20	190
3	16	115

Shipping cost for Factory (Rs)	1	2	2	6	10	5
	2	10	8	9	4	7
	3	5	6	4	3	8
Agency (j)		1	2	3	4	5
Demand		74	94	69	39	119
Sales price (Rs)		35	37	36	39	34

(14)

7. A small garment making unit has 5 tailors stitching 5 different types of garments. All 5 tailors are capable of stitching all the 5 types of garments. The output per day per tailor and the profit for each type of garment are given below:

i) Which type of garment should be assigned to which tailor in order to maximize profit assuming that there are no other constraints?

ii) If tailor *D* is absent for a specified period and no other substitute tailor is available, what should be the optimal assignment?

	GARMENTS				
TAILORS	1	2	3	4	5
A	7	9	4	8	6
B	4	9	5	7	8
C	8	5	2	9	8
D	6	5	8	10	10
E	7	8	10	9	9
Profit/ Garment	2	3	2	3	4

(14)

8. a) Find the optimal strategy of the players and the value of the game.

	B			
A	-3	4	2	9
	7	8	6	10
	6	2	4	-1

b) For what value of λ, the game with following pay-off matrix is strictly determinable?

		Player B		
		B_1	B_2	B_3
Player A	A_1	λ	6	2
	A_2	-1	λ	-7
	A_3	-2	4	λ

c) Reduce the given the game by dominance and find the game value.

		Player B				
		1	**2**	**3**	**4**	**5**
Player A	*I*	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

(3+4+7)

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