PH 561.1

Reg. No. [ ]

St. Aloysius College (Autonomous)

Semester I – P.G. Examination – M. Sc. Mathematics

February - 2022

ALGEBRA - I

Time : 3 Hours | Max. Marks : 70

Answer any **FIVE FULL** questions from the following. $(14 \times 5 = 70)$

1. (a) State and prove the Lagrange's theorem for finite groups.

   (b) Let x, y, z and w be elements of a group G with $xyz = 1$, where 1 is the identity. Does it follows that $yzx = 1$? Does it follows that $yxz = 1$?

   (c) Define a normal subgroup of a group. Show that a subgroup H of a group G is normal if and only if every left coset of H in G is also a right coset of H in G. (7+3+4)

2. (a) If H and K are normal subgroups of a group G with $G = HK$ and $H \cap K = \{e\}$, where e is the identity element in G, then show that G is isomorphic to $H \times K$.

   (b) Prove that every subgroup of a cyclic group is cyclic.

   (c) State and prove the first isomorphism theorem for groups. (6+4+4)

3. (a) Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be any function. Then prove that the following are equivalent:

      (i) ϕ is an isometry which fixes the origin.

      (ii) ϕ preserves the dot product.

      (iii) ϕ is an orthogonal linear operator.

   (b) Define an isometry of $\mathbb{R}^2$. Prove that every isometry of $\mathbb{R}^2$ is either a translation or a rotation or a reflection. (7+7)

4. (a) If p and q are prime numbers, then prove that no group of order pq is simple.

   (b) Prove that the group $GL_2(\mathbb{F}_2)$ of invertible matrices with mod 2 coefficients is isomorphic to the symmetric group S_3.

   (c) Show that a p-group G has a non-trivial center. (5+5+4)

5. (a) State and prove the first Sylow theorem for finite groups.

   (b) If p is a prime, then prove that every group of order p^2 is abelian. (9+5)

6. (a) If H and K are subgroups of a group G, then show that $[H : H \cap K] \leq [G : K]$.

   (b) State and prove the Cayley's theorem for finite groups.

Contd...2

(c) Determine all possible Class equations for groups of order 8. (6+5+3)

7. (a) Show that every group of order 15 is cyclic.

(b) Prove that every finite integral domain is a field.

(c) If $R \neq \{0\}$ is a commutative ring with identity having only two ideals $\{0\}$ and R itself, then prove that R is a field. (5+5+4)

8. (a) Let $\phi : R \to R'$ be a surjective ring homomorphism with kernel K. Prove that there is a bijective correspondence between the set of all ideals of R' and the set of all ideals of R that contain K.

(b) Describe all ring homomorphisms of $\mathbb{Z}$ into $\mathbb{Z} \times \mathbb{Z}$.

(c) Determine all units in the ring $\mathbb{Z}/12\mathbb{Z}$. (8+4+2)

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St Aloysius College (Autonomous)
Mangaluru
Semester I - P.G. Examinations - M.Sc. Mathematics
February 2022
LINEAR ALGEBRA-I

Time : 3 Hours | Max. Marks : 70

Answer any FIVE FULL questions from the following: $(14 \times 5 = 70)$

1. (a) Define a row-echelon matrix. Prove that any $m \times n$ matrix can be reduced to a row-echelon form by applying finitely many elementary row operations.

   (b) Solve the system of equations $AX = B$, when $A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 0 & 0 & 4 \\ 1 & -4 & -2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$

   (c) If A is an $n \times n$ invertible matrix, then show that $(A^{-1})^t = (A^t)^{-1}$. (7+5+2)

2. (a) Prove that the following conditions are equivalent for a square matrix A
   (i) A can be reduced to the identity matrix by applying elementary operations
   (ii) A is a product of elementary matrices
   (iii) A is invertible
   (iv) the system $AX = 0$ has only the trivial solution.

   (b) Derive the formula for the determinant of an $n \times n$ matrix in terms of permutations in S_n.

   (c) Determine the permutation matrix associated with the permutation $p = (1\ 3\ 2)(4\ 5) \in S_6$. (6+6+2)

3. (a) Define a basis for a vector space. Verify whether $\mathcal{B} = \{(2, 4, 0)^t, (4, 2, 4)^t, (3, 1, 1)^t\}$ is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

   (b) If S and L are finite subsets of a vector space V over a field F such that S spans V and L is linearly independent, then prove that $|L| \leq |S|$. Deduce that any two bases of a finite dimensional vector space have the same number of elements.

   (c) Determine the dimension of the space P_4 of all real polynomials of degree at most 4. (3+8+3)

4. (a) Prove that every vector space V of dimension n over a field F is isomorphic to the space F^n of column vectors.

   (b) Prove or disprove: $\mathbb{R}$ is a finite dimensional vector space over $\mathbb{Q}$.

   (c) Let V be an n-dimensional vector space and let $\mathcal{B}$ be an ordered basis of V. Prove that the collection of all ordered bases of V is $\{\mathcal{B}P : P \in GL_n(F)\}$. (6+2+6)

contd... 2

5. (a) If W_1 and W_2 are subspaces of a finite dimensional vector space over a field, then prove that $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

   (b) Show that the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices is a direct sum of the space of all $n \times n$ real symmetric matrices and the space of all $n \times n$ real skew-symmetric matrices.

(9+5)

6. (a) State and prove the rank-nullity theorem for a linear transformation on a finite-dimensional vector space.

   (b) Find a basis for the null space of $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T((x, y, z)') = \begin{bmatrix} x + y - z \\ x + z \end{bmatrix}$.

   (c) Compute the characteristic polynomial, eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 4 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 6 & 3 \end{bmatrix}$.

(6+3+5)

7. (a) Let T be a linear operator on an n-dimensional vector space V over a field F, whose characteristic polynomial has n distinct roots in F. Prove that there is an ordered basis of V with respect to which matrix of T is diagonal.

   (b) Do A and A^t have the same eigenvalues? the same eigenvectors? Justify.

   (c) Let A be a matrix in O_3 with $\det A = -1$. Prove that -1 is an eigenvalue of A.

(6+4+4)

8. (a) Define a rigid motion of $\mathbb{R}^n$. Show that a rigid motion is the composition of an orthogonal linear operator and a translation.

   (b) Compute e^A directly from the expansion, where $A = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. (11+3)

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St Aloysius College (Autonomous)

Mangaluru

Semester I - P.G. Examination - M.Sc. Mathematics

February - 2022

REAL ANALYSIS - I

Time: 3 hrs. Max Marks: 70

Answer any FIVE FULL questions from the following :

1. a) Define the notion of supremum for a bounded subset E of an ordered set S.

   b) State and prove Cauchy-Schwarz inequality.

   c) If $x \in \mathbb{R}$, $y \in \mathbb{R}$, $x > 0$, then prove that there exists a positive integer n such that $nx > y$ and hence show that there exists a rational number between any two real numbers. (2+5+7)

2. a) Define a countable set. Prove that every infinite subset of a countable set is countable.

   b) Let A be the set of all sequences of $0's$ and $1's$. Prove that A is uncountable.

   c) In a metric space, prove that any neighbourhood of a point is an open set.

   d) Let X denote the set of all non negative reals. Let $d: X \times X \to \mathbb{R}$ be defined by $d(x,y) = (x-y)^2, \forall x, y \in X$. Is d a metric on X? Justify your answer. (5+4+3+2)

3. a) Let X be a metric space and $E \subseteq X$. Then prove that E is open if and only if E^c is closed in X.

   b) Prove that every $k-cell$ in $\mathbb{R}^k$ is compact. (5+9)

4. a) Prove that every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

   b) Show that every non empty perfect set in $\mathbb{R}^k$ is uncountable.

   c) Define a connected set in a metric space. (5+7+2)

5. a) In a metric space prove that any convergent sequence converges to a unique point.

   b) If X is compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then show that $\{p_n\}$ converges in X.

   c) Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

   d) If $a > 0$ and p is real, then show that $\lim_{n\to\infty} \frac{n^p}{(1+a)^n} = 0$. (2+7+2+3)

Contd...2

6. a) Suppose $a_1 \geq a_2 \geq \ldots \geq 0$, then prove that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots\ldots$ converges.

b) Prove that 'e' is irrational.

c) State and prove the ratio test for convergence of series. (6+4+4)

7. a) Prove that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

b) Prove that any continuous mapping of a compact metric space X into a metric space Y is uniformly continuous. (7+7)

8. a) State and prove the Generalized Mean Value Theorem.

b) State and prove the Chain Rule for differentiation.

d) Let f be a real function defined on $[a, b]$ and differentiable at a point $x \in [a, b]$. Prove that f is continuous at x.

********* (6+6+2)

PS 564.1

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St. Aloysius College (Autonomous)

Semester I – P.G. Examination – M. Sc. Mathematics

February - 2022

GRAPH THEORY

Time : 3 Hours Max. Marks : 70

Answer any FIVE FULL questions from the following. (14 × 5 = 70)

1. (a) For any graph G with 6 points, show that either G or $\overline{G}$ contains a triangle.
   (b) Prove or disprove: if G is connected then $\overline{G}$ is disconnected.
   (c) Show that every cubic graph has an even number of points.
   (d) Define the following notions for a graph G: (i) walk (ii) geodesic (iii) girth (iv) square G^2 (v) circumference. (3+2+4+5)

2. Let G be a connected (p, q)-graph with $p \geq 4$. Prove that the intersection number $\omega(G) = q$ if and only if G has no triangles. (14)

3. (a) Prove that the following statements are equivalent for a point v of a connected graph $G = (V, E)$:
   (i) v is a cutpoint of G.
   (ii) There exist points u and w distinct from v in G such that v is on every $u - w$ path.
   (iii) There exists a partition of the set $V - \{v\}$ into subsets U and W such that for any points $u \in U$, $w \in W$, the point v is on every $u - w$ path.
   (b) Prove that every tree has a center consisting of either one point or two adjacent points. (6+8)

4. (a) Prove that the following statements are equivalent for a group G:
   (i) G is a tree.
   (ii) Any two points of G are joined by a unique path.
   (iii) G is connected and $p = q + 1$.
   (iv) G is acyclic and $p = q + 1$.
   (b) Determine the maximum number of cutpoints in a tree. (10+4)

5. (a) For any graph G, prove that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ denote the point connectivity, the line connectivity and the minimum degree of G, respectively.
   (b) Define the radius $r(G)$ and diameter $d(G)$ of a connected graph G and show that $r(G) \leq d(G) \leq 2r(G)$. (9+5)

Contd... 2

PS 564.1

6. (a) For a plane map G with p vertices, q edges and r faces, prove that $p - q + r = 2$.

   (b) Show that the graphs K_5 and $K_{3,3}$ are non-planar.

   (c) Show that every planar graph G with $p \geq 4$ has at least 4 points of degree not exceeding 5.

   (d) If G is a (p, q) maximal plane graph, then show that $q = 3p - 6$. (5+2+4+3)

7. (a) Define an Eulerian graph. Give an example of a graph which is both Eulerian and Hamiltonian.

   (b) Let G be a graph with $p \geq 3$ points. If for every n, $1 \leq n \leq \frac{p-1}{2}$, the number of points of degree not exceeding n is less than n and if, for odd p, the number of points of degree at most $\frac{p-1}{2}$ does not exceed $\frac{p-1}{2}$, then prove that G is Hamiltonian. (2+12)

8. (a) For any (p, q) graph G, prove that $\frac{p}{\beta_0} \leq \chi(G) \leq p - \beta_0 + 1$, where β_0 is the point independence number of G.

   (b) Prove that every planar graph is 5-colorable. (5+9)

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PS 566.1(R)

Reg No:

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Mangaluru
Semester I– P.G. Examination – M.Sc. Mathematics
February – 2022
OPERATIONS RESEARCH

Time: 3 hours **Max Marks: 70**

Answer any FIVE full questions.

1. a) Suppose that 8, 12 and 19 units of proteins, carbohydrates and fats are the minimum weekly requirement for a person. Suppose food A contains 2, 6 and 1 units of proteins, carbohydrates and fats respectively per kg and food B contains 1, 1 and 3 respectively per kg. If food A costs Rs. 85/kg and food B costs Rs. 40/kg, how many kilograms of each should be bought to minimize the cost and still meet the minimum requirements?

b) Solve the given LPP by graphical method:
Maximize $z = 3x_1 + 5x_2$
Subject to

$$2x_1 \leq 8$$
$$x_2 \leq 6$$
$$3x_1 + 2x_2 \leq 18$$
$$3x_1 + 5x_2 \geq 50$$
$$x_1, x_2 \geq 0$$

c) Write the standard form of the LPP
Maximize $z = 5x_1 + 10x_2 - 3x_3$
Subject to

$$2x_1 - 4x_2 + x_3 \leq 11$$
$$x_2 - x_3 \geq 8$$
$$3x_1 + 9x_3 \leq 1$$
$$x_1, x_2 \geq 0$$

(7+5+2)

2. A firm manufactures four different machines parts M_1, M_2, M_3 and M_4 made of copper and zinc. The amount of copper and zinc required for each machine part, their exact availability and profit earned from one unit of each machine part are given below.

	M_1 (kg)	M_2 (kg)	M_1 (kg)	M_1 (kg)	Exact availability
copper	5	4	2	1	100
zinc	2	3	8	1	75
Profit (Rs.)	12	8	14	10	

Solve the given problem using analytical method. Also, find

(i) Basic solutions
(ii) Basic feasible solutions
(iii) Non degenerate basic feasible solutions
(iv) Optimal basic feasible solutions

(14)

Contd...2

3. Write the simplex algorithm to obtain an optimal solution (if it exists) to an LPP. **(14)**

4. a) Solve the given LPP

Maximize $z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to $x_1 + 2x_2 + x_3 = 15$

$2x_1 + x_2 + 5x_3 = 20$

$x_1 + 2x_2 + x_3 + x_4 = 10$

$x_1, x_2, x_3, x_4 \geq 0$

b) Write the dual of the LPP

Maximize $z = 6x_1 + x_2 - 3x_3 + x_4$

Subject to $3x_1 + 2x_2 - 7x_3 - x_4 = 5$

$x_1 - 4x_2 + x_3 + 9x_4 = 7$

$x_1, x_2 \geq 0$

(12+2)

5. Give the transportation algorithm including the steps of the modified distribution method. **(14)**

6. A company has 3 factories manufacturing the same product and 5 sale agencies in different parts of the country. Production costs differ from factory to factory and sales price from agency to agency. Also the shipping cost per unit product from each factory to each agency is known. Given the following data find the production and distribution schedule most profitable to the company.

Factory (i)	Production cost /unit (Rs)	Maximum capacity (No. of units)
1	18	140
2	20	190
3	16	115

Shipping cost for Factory (Rs)	1	2	2	6	10	5
	2	10	8	9	4	7
	3	5	6	4	3	8
Agency (j)		1	2	3	4	5
Demand		74	94	69	39	119
Sales price (Rs)		35	37	36	39	34

(14)

Contd...3

7. A small garment making unit has 5 tailors stitching 5 different types of garments. All 5 tailors are capable of stitching all the 5 types of garments. The output per day per tailor and the profit for each type of garment are given below:

i) Which type of garment should be assigned to which tailor in order to maximize profit assuming that there are no other constraints?

ii) If tailor *A* is absent for a specified period and no other substitute tailor is available, what should be the optimal assignment?

	GARMENTS				
TAILORS	1	2	3	4	5
A	7	9	4	8	6
B	4	9	5	7	8
C	8	5	2	9	8
D	6	5	8	10	10
E	7	8	10	9	9
Profit/ Garment	3	5	7	3	2

(14)

8. a) Find the optimal strategy of the players and the value of the game.

	B			
A	-3	4	2	9
	7	8	6	10
	6	2	4	-1

b) For what value of λ, the game with following pay-off matrix is strictly determinable?

	Player B			
		B_1	B_2	B_3
Player A	A_1	λ	6	2
	A_2	-1	λ	-7
	A_3	-2	4	λ

c) Reduce the given the game by dominance and find the game value.

	Player B					
		1	**2**	**3**	**4**	**5**
Player A	*I*	1	3	2	7	4
	II	3	4	1	5	6
	III	6	5	7	6	5
	IV	2	0	6	3	1

(3+4+7)

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St Aloysius College (Autonomous)

Mangaluru

Semester I - P.G. Examination - M.Sc. Mathematics

February - 2022

ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 hrs. Max Marks: 70

Note: Answer any five full questions. (14x5=70)

1. a) If $\phi_1(t), \phi_2(t), \phi_3(t), \ldots\ldots, \phi_n(t)$ are solutions of the equation $x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots\ldots+a_n(t)x(t)=0$, then prove that they are linearly independent on I if and only if $W(\phi_1(t), \phi_2(t), \ldots\ldots, \phi_n(t)) \neq 0 \quad \forall t \in I$.

   b) State and prove Abel's formula for n^{th} order linear homogeneous differential equation.

(8+6)

2. a) Let $\phi_1, \phi_2, \phi_3, \ldots\ldots, \phi_n$ be n linearly independent solutions of the equation, $L_n(x) \equiv x^{(n)}(t)+b_1(t)x^{(n-1)}(t)+\ldots\ldots+b_n(t)x(t)=0$ existing on I. Let the real or complex valued function h be defined and continuous on I. Further assume that $W(t)=W(\phi_1(t), \phi_2(t), \cdots \phi_n(t))$ and $W_k(t)$ denote $W(t)$ with k^{th} column replaced by n elements $0, 0, 0, \cdots, 1$. Then prove that a particular solution $x_p(t)$ of $L_n(x)=h(t)$ is given by $x_p(t)=\sum_{k=1}^{n}\phi_k(t)\int_{t_0}^{t}\frac{W_k(s)\,h(s)}{W(s)}ds$; $t, t_0 \in I$.

   b) Compute the Wronskian of the two independent solutions of $x''-2tx'+2x=0$.

(8+6)

3. a) Prove that the functions x^4 and $|x|x^3$ are linearly independent on $[-1,-1]$ but they are linearly dependent on $[-1,0]$ and $[0,1]$.

   b) Solve $x^{(4)}+4x=2\sin t+1+3t^2+4e^t$. (6+8)

4. a) State and prove the orthogonal property of the Legendre polynomials.

   b) Find the Legendre series of the function $f(x)=e^x$. (8+6)

5. a) Derive Bessel's function of the first kind.

   b) With usual notations for Bessel functions show that:

      i) $\frac{d}{dt}\left(t^p J_p(t)\right)=t^p J_{p-1}(t)$

      ii) $\frac{d}{dt}\left(t^{-p} J_p(t)\right)=-t^{-p} J_{p+1}(t)$. (8+6)

Contd...2

I- Sem - 2022
Mathematics

6. a) Define a fundamental matrix of the system $X'(t) = A(t)X(t)$.

   b) Let $A(t)$ be an $n \times n$ continuous matrix valued function on I. Let $\Phi(t)$ be a fundamental matrix of the system $x' = A(t)x$, $t \in I$. Then show that

   $\det(\Phi(t))' = tr(A(t))\det(\Phi(t))$, $t \in I$.

   c) Let $\Phi(t)$, $t \in I$ be a fundamental matrix of the system, $x' = Ax$ where A is constant matrix such that $\Phi(0)$ is the identity matrix. Then show that

   $\Phi(t+s) = \Phi(t)\Phi(s)$, $\forall s,t \in I$. **(2+7+5)**

7. a) Show that the set of all solutions of the system $X'(t) = A(t)X(t)$, $t \in I$, forms an n-dimensional vector space over the field of complex numbers.

   b) Let $A(t)$ be an $n \times n$ continuous matrix on I and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $x'(t) = A(t)x(t)$, then show that $\Phi(t+\omega)$ is also a fundamental matrix. Justify that for any such $\Phi(t)$, there exists a periodic non-singular $P(t)$ with period ω and a constant matrix R such that $\Phi(t) = P(t)e^{tR}$.

   (7+7)

8. State and prove Picard's theorem. **(14)**

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