PH 561.2

Reg. No:

St Aloysius College (Autonomous)

Mangaluru

Semester II – P.G. Examination – M. Sc. Mathematics

April - 2018

ALGEBRA - II

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following (14x5=70)

1. a) Prove that every Euclidean domain is a principal ideal domain.

   b) Let R be an integral domain in which factorization into irreducible terminates. Show that R is a unique factorization domain if and only if every irreducible element of R is a prime element of R. (4+10)

2. a) If R is a unique factorization domain, then prove that $R[x]$ is a unique factorization domain.

   b) Is $\mathbb{Z}[x]$ a principal ideal domain? Justify. (12+2)

3. a) Let F be a field. Prove that all units in $F[x]$ are all non-zero elements of F.

   b) Verify whether the polynomial $x^5 - 64x^4 + 127x^3 - 200x + 99$ is irreducible in $\mathbb{Q}[x]$.

   c) State and prove Eisenstein's criterion for irreducibility of a polynomials in $R[x]$, where R is a unique factorization domain. (3+3+8)

4. a) Prove that every finite extension of a field is an algebraic extension. Is the converse true? Justify.

   b) If $F \subseteq K \subseteq L$ are fields such that, L is an algebraic extension of K and K is an algebraic extension of F, then prove that L is an algebraic extension of F.

   c) Find $[\mathbb{Q}(2+\sqrt[3]{2}, \sqrt[3]{3}) : \mathbb{Q}]$. (7+5+2)

5. a) Show that the set of all constructible real numbers form a subfield of $\mathbb{R}$ containing $\mathbb{Q}$.

   b) Prove that it is impossible to trisect the angle 60° using ruler and compass.

   c) If p is a prime number such that a regular p – gon can be constructed with ruler and compass, then show that $p = 2^r + 1$ for some integer $r \geq 0$. (6+4+4)

6. a) Let p be a prime and n be a positive integer. Then prove that there exists a field of order p^n.

Contd...2

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b) If K is a finite field, then show that K^{*}, the set of all non-zero elements of K is a cyclic group with respect to multiplication.

c) Determine the number of subfields of a field of order 5^{64}. (7+5+2)

7. a) Let F be a field of characteristic zero. Show that every finite extension of F is a simple extension.

b) Is every finite extension of a field, a Galois extension? Justify your answer.

c) If K is finite extension of a field F of characteristic zero, then show that $O(G(K/F)) \leq [K:F]$. (8+2+4)

8. a) If K is a Galois extension of a field F, then prove that the fixed field of K is F.

b) State and prove the fundamental theorem of Galois theory. (5+9)

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St Aloysius College (Autonomous)

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Semester II – P.G. Examination - M.Sc. Mathematics

April - 2018

NUMERICAL ANALYSIS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL Questions: **(5x14=70)**

1. a) Find a real root of the equation $logx - cosx = 0$ by the method of Regula-falsi, correct to three decimal places.

   b) Define order of an iteration method. Show that Newton-Raphson method is of second-order.

   c) Using Newton-Raphson method, find $\frac{1}{7}$. Carry out two iterations. **(6+6+2)**

2. a) Derive the Muller method to find the real root of the equation $f(x) = 0$.

   b) Find a real root of the polynomial equation $x^3 - 2x - 5 = 0$ by Birge-Vieta method. Carry out two iterations.

   c) State Gerschgorin theorem. **(7+5+2)**

3. a) Solve the equations

   $54x + y + z = 110$

   $2x + 15y + 6z = 72$

   $-x + 6y + 27z = 85$

   by Gauss-Seidel iteration method. Carry out four iterations by taking $(0,0,0)^t$ as the initial solution.

   b) Find the largest eigen-value and corresponding eigen-vector of the matrix

   $$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

   Carry out 5 iterations by taking $(1,1,1)^t$ as the initial eigen-vector. **(7+7)**

4. a) Derive Gregory-Newton's Backward interpolation formula. Also find the truncation error.

   b) Prove the existence and uniqueness of the interpolating polynomial of a function $f(x)$ at a given set of tabular points. **(7+7)**

5. a) Derive composite trapezoidal rule. Find its truncation error.

   b) Using the Newton-Cotes formula, derive Simpson's $\frac{1}{3}$rd rule.

   c) Find the linear interpolation formula for $f'(x)$ at $x = x_0$. **(6+5+3)**

Contd...2

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6. a) Evaluate the integral $\int_{-1}^{1} \frac{e^{-x^2}}{\sqrt{1-x^2}} dx$ using

   i) 2-point Gauss-Chebyshev quadrature

   ii) 3-point Gauss-Chebyshev quadrature

   b) Derive 3-point Gauss-Legendre quadrature formula.

   c) Evaluate the integral $\int_{y=1}^{1.5} \int_{x=1}^{2} \frac{dx\,dy}{x+y}$ using Simpson's rule with $h = 0.5$ along x-axis and $y = 0.25$ along y-axis.

(4+5+5)

7. a) Derive mid-point method to solve a first-order initial-value problem. Also find a local truncation error.

   b) Solve the IVP $y' = x + y$, $y(0) = 1$ on [0, 1] with $h = 0.2$ using Adam-Bashforth third order method.

(7+7)

8. a) Solve the IVP $y' = -2xy^2$, $y(0) = 1$, $h = 0.2$ on [0, 0.4] using Backward Euler method.

   b) Derive Dahlquist second order method to solve the BVP $y'' = f(x, y)$, $y(a) = \alpha$, $y(b) = \beta$ on $[a, b]$.

(6+8)

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St Aloysius College (Autonomous)
Mangaluru
Semester II – P.G. Examination - M. Sc. Mathematics
April - 2018

REAL ANALYSIS - II

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Define Riemann-Stieltje's integrability for a real bounded function f on an interval $[a, b]$ with respect to a monotonically increasing function α. If P^* is a refinement of a partition P of $[a, b]$, then prove that $L(P, f, \alpha) \leq L(P^*, f, \alpha)$.

   b) If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, prove that $fg \in \mathcal{R}(\alpha)$ on $[a, b]$.

   c) If f is continuous on $[a, b]$, then prove that $f \in \mathcal{R}$ on $[a, b]$.

   d) State and prove the Fundamental Theorem of Calculus. **(4+2+3+5)**

2. a) Let f be bounded real function defined on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$ $m \leq f(x) \leq M$ for all $x \in [a,b]$. Let ϕ be a continuous real function on $[m, M]$. If $h(x) = \phi(f(x))$ for all x in $[a, b]$, then prove that $h \in R(\alpha)$ on $[a, b]$.

   b) Define a rectifiable curve in $\mathbb{R}^n$. If a curve γ is continuously differentiable on $[a, b]$, then prove that γ is rectifiable and that $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$. **(6+8)**

3. a) Explain the terms uniform convergence and pointwise convergence of sequence of functions. State and prove Cauchy criterion for uniform convergence.

   b) Let $f_n(x) = n^2 x(1 - x^2)^n$, $0 \leq x \leq 1$, $n = 1, 2, 3, ...$ Find $\lim_{n \to \infty} f_n(x)$.

   c) Let $\mathcal{C}(X)$ denote the class of all complex valued, continuous, bounded functions on a compact metric space X. Prove that $\mathcal{C}(X)$ is a complete metric space with respect to the metric $\| f - g \| = sup\{| f(x) - g(x)| : x \in X\}$. **(5+3+6)**

4. a) Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then prove that $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and $f'(x) = \lim_{n \to \infty} f_n'(x)$ for all $x \in [a, b]$.

Contd...2

b) Define equicontinuous family of functions on a set E. If K is a compact metric space, $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then prove that $\{f_n\}$ is equicontinuous on K. (9+5)

5. State and prove Stone's generalization of Weierstrass theorem. (14)

6. a) Test the convergence of

i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ ii) $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

b) Let ϕ be bounded on $[a, \infty)$, integrable on $[a, t]$ for $t \geq a$. If $\int_a^\infty f(x)dx$ converges absolutely, then prove that $\int_a^\infty f(x)\phi(x)dx$ is convergent.

c) Examine the convergence of the integral $\int_a^\infty \frac{e^{-x}\sin x}{x^2}dx,\ a > 0.$ (4+7+3)

7. a) Suppose f maps an open subset E of $\mathbb{R}^n$ into $\mathbb{R}^m$. When do you say that f is differentiable in E?

b) Suppose f maps a convex open set $E \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, f is differentiable in E and there is a real number M such that $\|f'(x)\| \leq M$, for all $x \in E$. Prove that $\|f(b) - f(a)\| \leq M\|b-a\|$, for all $a, b \in E$. Hence prove that if $f'(x) = 0$ for all $x \in E$ then f is constant on E.

c) State and prove the contraction principle. (2+6+6)

8. State and prove the implicit function theorem. Illustrate it in the following case:

$n = 2,\quad m = 3,\quad f = (f_1, f_2)$ is a mapping from $\mathbb{R}^5$ to $\mathbb{R}^3$ given by

$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$

$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3$ and $a = (0, 1),\ b = (3, 2, 7)$ (14)

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St Aloysius College (Autonomous)
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Semester II – P.G. Examination - M. Sc. Mathematics
April - 2018

LINEAR ALGEBRA - II

Time: 3 Hours

Max. Marks: 70

Answer any **FIVE FULL** questions from the following (14x5=70)

1. a) Let V be a finite dimensional real vector space.
   i) Define a positive definite bilinear form on V. Give an example.
   ii) If V has a positive definite bilinear form, then prove that V has an orthonormal basis.

   b) Extend the vector $\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$ to an orthonormal basis of $\mathbb{R}^3$.

   c) Prove that a positive definite form is non-degenerate. (7+5+2)

2. State and prove the Sylvester's law for symmetric forms on a real vector space V. (14)

3. a) Prove the Spectral theorem for symmetric operators of Euclidean spaces.
   b) Prove that real symmetric matrix A is positive definite if and only if all its eigen values are positive. (9+5)

4. a) If V is a vector space of dimension m over a field F with non-degenerate skew-symmetric form $\langle\ ,\ \rangle$, then prove that m is even and there is a basis of V such that matrix of $\langle\ ,\ \rangle$ with respect to this basis is $J_{2n}=\begin{bmatrix}0 & I_n\\ -I_n & 0\end{bmatrix}$ where $n=\frac{m}{2}$.
   b) If S is a real, skew-symmetric matrix and $I+S$ is invertible, prove that $(I-S)(I+S)^{-1}$ is orthogonal. (12+2)

5. a) Let M be an additive abelian group. Show that there is only one way of making M as a $\mathbb{Z}$-module. Further, if M is finite, then is it true that M is a free $\mathbb{Z}$-module? Justify.
   b) Let $R=\mathbb{C}[X,Y]$ and M be an ideal generated by X and Y. Is M a free R-module? Justify your answer.
   c) Prove that any two bases of a free R-module M have the same cardinality, provided R is a non-zero ring. (4+6+4)

Contd...2

6. a) If A is an $m \times n$ integer matrix, prove that there exist products P, Q of elementary integer matrices such that QAP^{-1} is diagonal.

   b) If $\phi : V \to W$ is a homomorphism of free abelian groups then show that there exist bases of V and W such that the matrix of the homomorphism ϕ has diagonal form.

(9+5)

7. a) Define a Noetherian ring. If M is a finitely generated module over a Noetherian ring R, then prove that every submodule of M is finitely generated.

   b) Prove that a finitely generated module M over a Noethrian ring R has a presentation matrix.

   c) Identify the abelian group which has $\begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix}$ as a presentation matrix.

(8+4+2)

8. Prove the structure theorem for finitely generated abelian groups.

(14)

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Mangaluru

Semester II – P.G. Examination - M.Sc. Mathematics

April - 2019

ALGEBRA II

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL Questions: **(5x14=70)**

1. a) Prove that every Euclidean domain is principal ideal domain (PID). Give an example of PID which is not a Euclidean domain.

   b) Prove that every principal ideal domain is a unique factorization domain (UFD). **(5+9)**

2. a) If $f(x)$ and $g(x)$ are primitive polynomials in $\mathbb{Z}[x]$ then prove that, their product is also primitive.

   b) Prove that $\mathbb{Z}[x]$ is a unique factorization domain. **(5+9)**

3. a) Let $f(x) = a_n x^n + \ldots + a_1 x + a_0$ be an integer polynomial and p be a prime integer such that $p \nmid a_n$. If the residue $\overline{f}$ of f modulo p is an irreducible element in $\mathbb{F}_p[x]$, then prove that f is irreducible element in $\mathbb{Q}[x]$.

   b) Verify whether the polynomial $x^2 + x + 1$ is irreducible in $\mathbb{Q}[x]$.

   c) Find $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})]$ **(8+3+3)**

4. a) Let K be an extension of field F and $\alpha, \beta \in K$ be algebraic over F. Then prove that there exists an F-isomorphism from $F(\alpha)$ to $F(\beta)$ which sends α to β if and only if α and β have the same minimal polynomial in $F[x]$.

   b) Prove or disprove the following:

      i) Every finite extension of a field F is an algebraic extension.

      ii) Every algebraic extension is a finite extension. **(8+6)**

5. a) If $a > 0$ is constructible real number, then prove that $\sqrt{a}$ is also constructible.

   b) Decide whether or not the regular 9-gon is constructible by ruler and compass.

   c) Let p be a prime and n be a positive integer. Prove that there exists a unique field of order p^n up to isomorphism. **(4+2+8)**

6. a) Define an algebraically closed field. Prove that no finite extension of $\mathbb{Q}$ is algebraically closed.

Contd...2

b) State and prove fundamental theorem of Algebra.

(4+10)

7. a) Prove that every finite extension of a field of characteristic zero has a primitive element.

b) Find the splitting field and degree of the splitting field of $x^4 - 2$ over $\mathbb{Q}$.

(9+5)

8. a) If K is a finite extension of a field F of characteristic zero, then show that $O(G(K/F)) \leq [K:F]$.

b) Let $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Prove that K is Galois extension of $\mathbb{Q}$. Determine its Galois group.

(8+6)

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Semester II – P.G. Examination - M. Sc. Mathematics
April - 2019
NUMERICAL ANALYSIS

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following (14x5=70)

1. a) Apply Newton-Raphson's method to determine a root of the equation $\cos x - xe^x = 0$. Carry out four iterations.

   b) Derive the Regula-Falsi method to find the root of an equation $f(x)=0$.

   c) Find a real root of $2x - \log_{10} x = 7$ correct to 3 decimal places using iteration method. (6+5+3)

2. a) Using Muller's method find a root of the equation $x^3 - x - 1 = 0$.

   b) Apply Gauss-Jordan method to solve the equations

   $$x + y + z = 9$$
   $$2x - 3y + 4z = 13$$
   $$3x + 4y + 5z = 40$$

   c) Find the largest eigen-value and the corresponding eigen-vector of the matrix $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ carry out 3 iterations by taking $(1,1,1)'$ as the initial eigen vector. (4+5+5)

3. a) Use synthetic division and perform two iterations by Birge-Vieta method to find the smallest positive root of the equation $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0$.

   b) Derive the Chebyshev method to find the real root of the equation $f(x)=0$. (7+7)

4. a) From the following table of values of x and y, obtain $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.0$

x	1.0	1.2	1.4
y	2.718	3.326	4.055

   b) Evaluate $\int_{-2}^{2}\int_{0}^{4}(x^2 - xy + y^2)\,dx\,dy$, $h = k = 2$ using the trapezoidal rule.

   c) Derive one-point Gauss-Legendre formula. (6+5+3)

Contd...2

5. a) Derive Gregory Newton's forward interpolation formula.

b) The pressure β of wind corresponding to velocity v is given by the following data. Estimate β when $v = 25$.

v	10	20	30	40
β	1.1	2.0	4.4	7.9

(7+7)

6. a) Derive Hermite's interpolation formula for the data points $(x_i, y_i), (x_i, y_i') \quad i = 0,1,\ldots,n$.

b) Obtain the least square polynomial approximation of degree one and two for $f(x) = \sqrt{x}$ on $[0,1]$.

(6+8)

7. a) Using Runge-Kutta method of order four, find the solution of y at $x = 0.2$ for the IVP $\frac{dy}{dx} = y - x, \quad y(0) = 2$. Take $h = 0.1$

b) Derive Adams Moulton method to solve the Initial Value Problem. (7+7)

8. a) Use finite difference method to solve the Boundary Value Problem defined by $y'' + xy' - 2y = 0 \quad y(0) = 1, \quad y(1) = 2$ take $h = \frac{1}{4}$.

b) From the Taylor series for the initial value problem $y'(t) = y(t), \; y(0) = 1$ find $y(0.1)$. Also show that the truncation error satisfies

$$|\tilde{y}(x_i) - y_i| \le \frac{h^n}{(n+1)!} x_i e^{x_i}.$$

(7+7)

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St Aloysius College (Autonomous)

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Semester II – P.G. Examination - M. Sc. Mathematics

April - 2019

REAL ANALYSIS - II

Time: 3 Hours

Max. Marks: 70

Answer any FIVE FULL questions from the following (14x5=70)

1. a) Let f be a bounded real function defined on $[a, b]$. Define the Riemann integral of f over $[a, b]$.

   b) If $f, g \in \mathcal{R}(\alpha)$ on $[a, b]$, prove that $f+g \in \mathcal{R}(\alpha)$ on $[a, b]$.

   c) If f is monotonic on $[a, b]$, prove that $f \in \mathcal{R}$ on $[a, b]$.

   d) State and prove the Fundamental Theorem of Calculus. (3+3+3+5)

2. a) Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Prove that $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

   b) Define a rectifiable curve in $\mathbb{R}^n$. If a curve γ is continuously differentiable on $[a, b]$, prove that γ is rectifiable and that

   $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,dt$. (7+7)

3. a) Define the notions of pointwise convergence and uniform convergence of a sequence $\{f_n\}$ of functions. State and prove Cauchy criterion for uniform convergence of a sequence of functions.

   b) Let $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ $\forall x \in \mathbb{R}$, $n = 1, 2, 3, \ldots$ Prove that $\{f_n\}$ converges to a function f but $\{f_n'\}$ does not converges to f'.

   c) Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then prove that sequence $\{f_n\}$ converges uniformly on $[a, b]$. Further if $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a,b]$, prove that

   $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a, b]$. (5+2+7)

4. a) Suppose K is compact, $\{f_n\}$ is a sequence of continuous functions that converge pointwise to a continuous function f on K and $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1,2,3,\ldots$. Prove that $f_n \to f$ uniformly on K.

   b) Prove that there exists a real continuus function on the real line which is nowhere differentiable. (7+7)

Contd...2

5. a) Define equicontinuous family of functions on a set E. If K is a compact metric space, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \ldots$, and if $\{f_n\}$ converges uniformly on K, then prove that the family $\{f_n : n \geq 1\}$ is equicontinuous on K.

   b) If f is a continuous complex function on $[a, b]$, prove that there exists a sequence $\{P_n\}$ of polynomials such that $P_n \to f$ uniformly on $[a, b]$.

(5+9)

6. a) Test the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$.

   b) Prove that every absolutely convergent integral is convergent.

   c) Let ϕ be bounded on $[a, \infty)$, integrable on $[a, t]$ for $t \geq a$. Let $\int_a^\infty f(x)dx$ converge absolutely at ∞. Prove that $\int_a^\infty f(x)\phi(x)dx$ is absolutely convergent at ∞.

(2+5+7)

7. a) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, when do you say that f is differentiable in E?

   b) If $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0) \end{cases}$

   Prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of $\mathbb{R}^2$ and f is not continuous at $(0, 0)$

   c) Suppose E is an open set in $\mathbb{R}^n$, f maps E into $\mathbb{R}^m$, f is differentiable at $x_0 \in E$, g maps an open set containing $f(E)$ into $\mathbb{R}^k$ and g is differentiable at $f(x_0)$. Then prove that the mapping F of E into $\mathbb{R}^k$ defined by $F(x) = g(f(x))$, for all $x \in E$ is differentiable at x_0 and $F'(x_0) = g'(f(x_0))\, f'(x_0)$.

(2+4+8)

8. a) State and prove the contraction principle.

   b) State and prove the inverse function theorem.

(7+7)

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Semester II – P.G. Examination - M. Sc. Mathematics

April - 2019

LINEAR ALGEBRA - II

Time: 3 Hours Max. Marks: 70

Answer any **FIVE FULL** questions from the following (14x5=70)

1. a) If A is the matrix of a bilinear form with respect to a basis, then prove that the matrices A' which represents the same form with respect to different bases are the matrices $A' = QAQ^t$ for some invertible matrix Q.

   b) If V is a finite dimensional vector space with positive definite bilinear form, then show that V has an orthonormal basis.

   c) Find an orthonormal basis for $\mathbb{R}^2$ with respect to the form X^tAY, where $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$.

(4+6+4)

2. State and prove Sylvester's law of symmetric form on a real vector space.

(14)

3. a) Prove that eigenvalues of symmetric operator are real.

   b) State and prove the spectral theorem for symmetric operators.

   c) If A is a real symmetric matrix then show that there exists an orthogonal matrix P such that $PAP^t = D$, where D is a real diagonal matrix.

(2+8+4)

4. a) Show that a real symmetric $n \times n$ matrix A is positive definite if $\det A_i$ is positive for all i, $1 \le i \le n$ where A_i is the upper left $i \times i$ submatrix of A.

   b) If V is a finite dimensional complex vector space with hermitian form $\langle \, , \rangle$ and V has an orthonormal basis, then prove that $\langle \, , \rangle$ is positive definite.

   c) Show that the matrix relating two orthonormal bases of a hermitian space is unitary.

(7+3+4)

5. a) Prove that any two bases of the same free module over a ring R have the same cardinality.

   b) Prove or disprove the following:

      i) A submodule of a free module is free

      ii) Every linearly independent set in a free module can be extended to a basis.

(7+7)

Contd...2

6. If V is a finitely generated module over a Noetherian ring, then prove the following:

   i) Every submodule of V is finitely generated.

   ii) V has a presentation matrix.

(9+5)

7. a) If A is an $m \times n$ integer matrix, prove that there exist products P, Q of elementary integer matrices such that QAP^{-1} is diagonal.

   b) If $\phi : V \to W$ is a homomorphism of free abelian groups, then prove that there exist bases of V and W such that the matrix of ϕ has the diagonal form.

(9+5)

8. State and prove the structure theorem for abelian groups. (14)

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