PH 561.3

Reg. No. | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester III– P.G. Examination – M.Sc. Mathematics
November - 2019
COMPLEX ANALYSIS - I

Time: 3 hrs. Max Marks: 70

Answer any FIVE FULL questions. (14x5=70)

1. a) In the spherical representation of the extended complex plane, show that circles and straight lines in the extended complex plane correspond to circles in the Riemann sphere.
   b) Find the distance between the stereographic images of $2+i$ and $1-i$. **(9+5)**
2. a) State and prove Cauchy's inequality.
   b) Show that $\left|\frac{z_1-z_2}{1-\overline{z_1}\,z_2}\right| < 1$ if $|z_1|<1$ and $|z_2|<1$.
   c) Find the values of $\sqrt[4]{-1}$. **(7+4+3)**
3. a) Prove that the function $u+iv$ determined by a pair of conjugate harmonic functions is always analytic.
   b) Prove that if all the zeros of a polynomial $P(z)$ lie in a half-plane, then all the zeros of the derivative $P'(z)$ also lie in the same half-plane. **(7+7)**
4. a) Define the exponential function e^z and show that e^{iz} has a least positive period 2π and all other periods are integer multiples of 2π.
   b) Show that $e^{a+b} = e^a.e^b \quad \forall\, a,b \in \mathbb{C}$. **(10+4)**
5. a) If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then show that there exists R with $0 \le R \le \infty$ such that the series converges absolutely for every z with $|z|<R$, the sum of the series is an analytic function in $|z|<R$, and the sum diverges for $|z|>R$.
   b) Show that if $f(z)=u(z)+iv(z)$ is an analytic function in a region Ω and $|f(z)|$ is constant in Ω, then $f(z)$ must be a constant function. **(10+4)**
6. a) If $p(x,y)$ and $q(x,y)$ are real or complex valued continuous functions defined in a region Ω and if γ is any curve in Ω, then show that $\int_\gamma pdx+qdy$ depends only on the end points of γ if and only if there exists a function $u(x,y)$ in Ω with $\frac{\partial u}{\partial x}=p$ and $\frac{\partial u}{\partial y}=q$.
   b) Show that linear transformation carries circles into circles. **(10+4)**
7. a) State and prove Cauchy's theorem for a rectangle.
   b) Use Cauchy's integral formula to evaluate $\int_C \frac{1}{z(z-1)}\,dz$ where C is the circle $|z|=3$. **(9+5)**
8. a) If $\varphi(\xi)$ is continuous on an arc γ, show that $F_n(z)=\int_\gamma \frac{\varphi(\xi)}{(\xi-z)^n}\,d\xi$ is analytic in each of the regions determined by γ and its derivative is $F_n'(z)=nF_{n+1}(z)$.
   b) Find a necessary and sufficient condition for an isolated singularity of $f(z)$ to be a removable singularity. **(10+4)**

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Mangaluru

Semester III– P.G. Examination – M.Sc. Mathematics

November - 2019

TOPOLOGY

Time: 3 hrs. Max Marks: 70

Answer any FIVE FULL questions from the following: (14x5=70)

1. a) Define basis for a topology on a nonempty set X. Show that the collection of all open intervals in $\mathbb{R}$ is a basis for a topology on $\mathbb{R}$.

   b) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies τ and τ' respectively on a set X, then show that τ' is finer than τ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

   c) Define the topologies of $\mathbb{R}, \mathbb{R}_l$ and $\mathbb{R}_k$ and compare them. **(3+5+6)**

2. a) Define subspace topology. Prove that if Y is a subspace of X, then $A \subseteq Y$ is closed in Y if and only if $A = C \cap Y$ where C closed in X.

   b) Define the closure $\bar{A}$ of a subset A of a topological space X. Prove that $x \in \bar{A}$ if and only if every neighbourhood of x intersects A.

   c) If A is a subset of a topological space X then prove that $\bar{A} = A \cup A'$ where A' is the set of all limit points of A. **(4+6+4)**

3. a) Define a Hausdorff space. Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x,x) \in X \times X / x \in X\}$ is closed in $X \times X$.

   b) Prove that a subspace of a Hausdorff space is Hausdorff.

   c) If X is a topological space, then prove that X is T_1 if and only if every finite set in X is closed. **(6+2+6)**

4. a) State and prove Pasting Lemma. Show that if $f, g: \mathbb{R} \to \mathbb{R}$ are continuous, then the map $h: \mathbb{R} \to \mathbb{R}$ defined by $h(x) = min\{f(x), g(x)\}$ is continuous.

   b) Define open maps and closed maps. Show that a continuous open map need not be closed. **(6+8)**

5. a) Prove that the union of a collection of connected subspaces of X that have a point in common is connected.

   b) If A is a connected subspace of X, then prove that $\bar{A}$ is connected.

   c) Prove that the product of two connected spaces is connected.

   d) If $f: [0,1] \to [0,1]$ is a continuous map, then show that there exists a point $x \in [0,1]$ such that $f(x) = x$. **(4+4+4+2)**

6. If $f: X \to Y$ is a bijective continuous map where X is compact and Y is Hausdorff, then prove that f is a homeomorphism. **(14)**

7. a) Define a second countable space. If X is second countable show that every open cover of X contains a countable subcollection covering X.

   b) Define a regular space. Prove that every compact Hausdorff space is regular.

   c) Prove that a closed subspace of a normal space is normal. **(4+7+3)**

8. State and prove Urysohn's Lemma. **(14)**

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St Aloysius College (Autonomous)
Mangaluru
Semester III - P.G. Examination - M.Sc. Mathematics
November - 2019
ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 Hours Max.Marks:70

PART -A

Answer any **FIVE** full questions.

1. a) If ϕ_1 is a solution of $x''+a_1(t)x'+a_2(t)x=0$ on an interval I and if $\phi_1(t) \neq 0$, $\forall t \in I$.

   Then show that $\phi_2(t) = \phi_1(t) \int_{t_0}^{t} \frac{1}{\phi_1^2(s)} \exp\left(\int_{t_0}^{s} -a_1(\xi)\,d\xi \right) ds$ is another solution.

   Further show that these solutions are linearly independent on I.

   b) Using the method of undetermined coefficients, solve the differential equation $x'' - 2x' + x = t + e^t$.

(8+6)

2. a) If $\phi_1(t), \phi_2(t) \dots, \phi_n(t)$ are solutions of equation $x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)\,x(t) = 0$, then prove that they are linearly independent on I if and only of $W(\phi_1(t), \phi_2(t), \dots \phi_n(t)) \neq 0 \ \ \forall t \in I$.

   b) Solve the IVP: $x'' - 3x' + 2x = 0$, $x(0) = -1$, $x'(0) = 0$.

   c) Find a particular solution of $x'' + x' = \sin 2t$.

(8+4+2)

3. a) Find two linearly independent solutions of $x'' + 2x' + x = 0$.

   b) State and prove Abel's formula for n^{th} order linear homogeneous differential equation.

   c) Compute the Wronskian of t^2 *and* $t|t|$ on $\mathbb{R}$ and show that they are linearly independent on $\mathbb{R}$.

(2+8+4)

4. a) Find the Legendre series of the function $f(x) = e^x$.

   b) Show that the Legendre polynomial $P_n(x)$ satisfies $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$.

   c) Define regular singular point with an example.

(6+6+2)

5. a). Derive Bessel's function of the First kind.

   b) With usual notations for Bessel functions show that

   (i) $\frac{d}{dt}\left(t^P J_P(t)\right) = t^P J_{P-1}(t)$

   (ii) $\frac{d}{dt}\left(t^{-P} J_P(t)\right) = -t^{-P} J_{P+1}(t)$

(9+5)

Contd...2

6 a) Let $A(t)$ be an $n \times n$ continuous matrix on I. Let $\Phi(t)$ be an $n \times n$ matrix whose i^{th} column $(i = 1, 2, ..., n)$ is a solution of $X' = A(t)X, t \in I$. Then show that $W(t) = \det \Phi(t)$ satisfies $W'(t) = tr(A(t))\, W(t),\ t \in I$. If $t_0 \in I$ then show that $W(t) = W(t_0)\ exp \int_{t_0}^{t} tr(A(s))ds,\ t \in I$.

b) Find the fundamental matrix for $X' = A(t)X$ where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

(7+7)

7 a). Let $A(t)$ be an $n \times n$ continuous matrix on $(-\infty, \infty)$ and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $X' = A(t)X$, then show that $\Phi(t+\omega)$ is also a fundamental matrix. Show that any fundamental matrix $\Phi(t)$ can be written as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a non-singular periodic matrix of period ω and R is a constant matrix.

b) If A is an $n \times n$ constant matrix, find a fundamental matrix of $X' = A(t)X$, by considering the cases where the eigen values of A are all distinct.

(7+7)

8 a) State and prove Gronwall's inequality.

b) Find the first four Picard's approximations of the initial value problem $x' = t + x,\ x(0) = 1$. Further find the limit of these approximations.

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(7+7)

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St Aloysius College (Autonomous)
Mangaluru
Semester III - P.G. Examination - M.Sc. Mathematics
November - 2019
COMMUTATIVE ALGEBRA

Time: 3 Hours

Max.Marks:70

PART -A

Answer any FIVE full questions.

1. a) Define a nilpotent element in a ring A. Show that the set of all nilpotent elements forms as ideal of A.

b) Define the Jacobson radical $\mathcal{J}$ of a ring A. Prove that an element $x \in \mathcal{J}$ if and only of $1 - xy$ is a unit in A, for every $y \in A$

c) Prove that in the polynomial ring $A[x]$, the nilradical ad the Jacobson radical are equal.

(5+5+4)

2 a) Prove that every non-unit in a non-zero ring A is contained in a maximal ideal.

b) Let A be a Boolean ring. Prove that every prime ideal $\mathfrak{p}$ in A is maximal and that $A/\mathfrak{p}$ is a field with two elements.

c) If f : A→B is a ring homomorphism, define the extension I^e of an ideal I of A. If I is a prime ideal of A, is I^e prime? Justify your answer.

(7+4+3)

3. a) If $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ are prime ideals of a ring A, and I is an ideal of A, contained in $\bigcup_{j=1}^{n} \mathfrak{p}_j$, then prove that $I \subseteq \mathfrak{p}_j$ for some j, $1 \le j \le n$.

b) If A is a ring, describe the prime spectrum of A. Further, show that spectrum A is a compact space.

(6+8)

4. a) Define a finitely generated A- module. If $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules such that M' and M'' are finitely generated, then show that M is finitely generated.

b) Let M be a finitely generated A-module, let I be an ideal of A and φ be an A-module endomorphism of M such that $\varphi(M) \subseteq IM$. Prove that φ satisfies an equation of the form $\varphi^n + a_1\varphi^{n-1} + \cdots + a_{n-1}\varphi + a_n = 0$, where $a_i \in I, 1 \le i \le n$. Hence deduce the Nakayama's lemma.

(5+9)

5 a). Let I be an ideal of a ring A and let $S = 1 + I$. Show that S is multiplicatively closed in A. Deduce that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$.

Contd...2

b) If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A- modules, show that for a multiplicatively closed set S in A, $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact.

c) If I, J are ideals of a ring A such that J is finitely generated, show that $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$. **(4+5+5)**

6 a) Show that $A_\mathfrak{p}$ is a local ring, where $\mathfrak{p}$ is a prime ideal of A.

b) If $\varphi: M \to N$ is as A-module homomorphism, show that the following statements are equivalent:

i) φ is injective

ii) $\varphi_\mathfrak{p}: M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for every prime ideal $\mathfrak{p}$ of A.

iii) $\varphi_\mathfrak{m}: M_\mathfrak{m} \to N_\mathfrak{m}$ is injective for every maximal ideal $\mathfrak{m}$ of A.

c) Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists an element $s \in S$ such that $sM = 0$. **(4+5+5)**

7 a). Let S be a multiplicatively closed set in ring A. Prove that there exists a one-to-one correspondence between the set of all prime ideals of $S^{-1}A$ and the set of all prime ideals of A disjoint with S.

b) Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{p}$ be a prime ideal of A. Prove that $\mathfrak{p}$ is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

(8+6)

8 a) Define a minimal primary decomposition. State and prove the First Uniqueness Theorem.

b) In a Noetherian ring, prove that every irreducible ideal is primary.

(8+6)

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St Aloysius College (Autonomous)

Mangaluru

Semester III – P.G. Examination - M. Sc. Mathematics

November - 2018

COMPLEX ANALYSIS I

Time: 3 Hours | Max. Marks: 70

Answer any **FIVE FULL** questions from the following (14x5=70)

1. a) Find the value of $\sqrt{1+i}$.

   b) In the spherical representation of the extended complex plane, show that the circles and straight lines in the extended complex plane correspond to circles on the Riemann sphere.

   c) Find the value of $\left|\frac{(4-5i)^{658}}{(5+4i)^{658}}\right|$. (2+10+2)

2. a) Prove that the function $f(z)=|z|^2$ is continuous everywhere but nowhere differentiable except at 0.

   b) State and prove a necessary and sufficient condition for a function $f(z)=u(z)+i\,v(z)$ to be analytic in a region Ω.

   c) Show that e^z has no zeros in the complex plane.

   (3+9+2)

3. a) Determine all the values of i^i.

   b) State and prove Abel's limit theorem.

   c) Find the radius of convergence of $\sum_{n=0}^{\infty}\frac{z^n}{n!}$. (2+10+2)

4. a) If z_1, z_2, z_3 are three distinct points of the extended complex plane, define the cross ratio (z_1, z_2, z_3, z_4) and show that $(Tz_1, Tz_2, Tz_3, Tz_4)=(z_1, z_2, z_3, z_4)$ if T is a linear fractional transformation.

   b) Find the bilinear transformation which maps $-1, 0, 1$ of the z-plane onto $-1, -i, 1$ of the w-plane.

   c) Find the image of the circle $|z-3i|=3$ under the map $w=\frac{1}{z}$

   (8+3+3)

Contd...2

5. a) If $p(x,y)$ and $q(x,y)$ are real or complex valued continuous functions defined in a region Ω and if γ is any curve in Ω, then show that $\int_\gamma pdx + qdy$ depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

b) Prove that $\int_C \frac{dz}{z-a} = 2\pi i$ where C is a circle with center 'a'. (10+4)

6. a) State and prove Cauchy's theorem for a rectangle.

b) Show that the index of 'a' with respect to a piecewise differentiable closed curve γ which does not pass through 'a' is constant in each of the regions determined by γ. (10+4)

7. a) State and prove Cauchy's integral formula.

b) Find the value of $\int_{|z|=1} \frac{e^z}{z-\pi} dz$.

c) State and prove Liouville's theorem. (7+2+5)

8. a) State the maximum principle for analytic functions. If $f(z)$ is continuous on a closed and bounded set E and analytic in the interior of E, then show that the maximum of $|f(z)|$ on E is assumed on the boundary of E.

b) If $f(z)$ is analytic in a region Ω containing 'a', then show that it is possible to write

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + f_n(z)(z-a)^n,$$ where

$f_n(z)$ is analytic in Ω. (7+7)

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St Aloysius College (Autonomous)

Mangaluru

Semester III – P.G. Examination - M. Sc. Mathematics

November - 2018

TOPOLOGY

Time: 3 Hours

Max. Marks: 70

Answer any <u>FIVE FULL</u> questions from the following **(14x5=70)**

1. a) Define basis for topology and the topology generated by a basis. Show that the collection τ generated by a basis $\mathscr{B}$ is a topology on X.

   b) If $\mathscr{B}$ is a basis for some topology on X then prove that the collection of all unions of members of $\mathscr{B}$ is a topology on X.

   c) If $\mathfrak{C}$ is a collection of open subsets of a topological space X such that for each $x \in X$ and each open set U of X containing x, there is an element C in $\mathfrak{C}$ such that $x \in C \subset U$ then show that $\mathfrak{C}$ is a basis for the topology on X. **(5+4+5)**

2. a) Define closure $\overline{A}$ of a subset A of a topological space X. Prove that $x \in \overline{A}$ if and only if every neighbourhood of x intersects A.

   b) Define the interior and boundary of a subset A of a topological space X. Show that $\overline{A}$ is the disjoint union of interior and boundary of A. **(6+8)**

3. a) Prove that every T_2 - space is a T_1 - space. Discuss the converse.

   b) Prove that the product of two T_2 - spaces is a T_2-space.

   c) If X is a topological space then prove that X is T_1 if and only if every finite set in X is closed. **(4+4+6)**

4. a) Let X and Y be topological spaces and $f: X \to Y$ be a map. Show that the following are equivalent:

   i) f is continuous

   ii) for every subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$

   iii) for every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

   b) State and prove Pasting Lemma. Show that if $f, g: \mathbb{R} \to \mathbb{R}$ are continuous, then the map $h: \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \min\{f(x), g(x)\}$ is continuous.

   c) Give an example to show that a one-to-one continuous mapping of one topological space onto another need not be a homeomorphism. **(6+6+2)**

Contd...2

5. a) Show that the union of a collection of connected sets, in a topological space, having a point in common is connected.
   b) Is $\mathbb{R}_l$ a connected topological space? Justify.
   c) Prove that finite cartesian product of connected spaces is connected.
   d) Prove that a space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(4+1+5+4)

6. a) Define a compact space. Is $\mathbb{R}$ compact? Justify.
   b) Prove that every closed subspace of a compact space is compact.
   c) Show that continuous image of a compact space is compact.
   d) Show that a topological space is compact if and only if for every collection $\mathfrak{C}$ of closed sets in X satisfying the finite intersection property, the intersection $\bigcap_{C \in \mathfrak{C}} C$ is non-empty.

(2+4+4+4)

7. a) Define a second countable space. If X is second countable show that every open cover of X contains a countable sub collection covering X.
   b) Prove that every compact T_2 - space is a T_4 - space.
   c) Give an example of a metrizable space which is not second countable.

(4+8+2)

8. State and prove Urysohn Lemma.

(14)

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St Aloysius College (Autonomous)
Mangaluru
Semester III - P.G. Examination - M.Sc. Mathematics
November - 2018
ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 Hours

Max.Marks:70

PART -A

Answer any __FIVE__ full questions.

1. a) If ϕ_1 is a solution of $x''+a_1(t)x'+a_2(t)x=0$ on an interval I and if $\phi_1(t) \neq 0,\ \forall t \in I$.

   Then show that $\phi_2(t) = \phi_1(t) \int_{t_0}^{t} \frac{1}{\phi_1^2(s)} \exp\left(\int_{t_0}^{s} -a_1(\xi)\,d\xi\right) ds$ is another solution.

   Further show that these solutions are linearly independent on I.

   b) Using the method of undetermined coefficients, solve the differential equation $x'' - 2x' + x = t + e^t$. **(8+6)**

2. a) If $\phi_1(t), \phi_2(t) \ldots, \phi_n(t)$ are solutions of equation $x^{(n)}(t) + a_1(t)x^{(n-1)}(t) + \cdots + a_n(t)\,x(t) = 0$, then prove that they are linearly independent on I if and only if $W(\phi_1(t), \phi_2(t), \ldots \phi_n(t)) \neq 0\ \ \forall t \in I$.

   b) Solve the IVP: $x'' - 3x' + 2x = 0,\ \ x(0) = -1,\ \ x'(0) = 0$.

   c) Find a particular solution of $x'' + x' = \sin 2t$. **(8+4+2)**

3. a) Find two linearly independent solutions of $x'' + 2x' + x = 0$.

   b) State and prove Abel's formula for n^{th} order linear homogeneous differential equation.

   c) Compute the Wronskian of t^2 *and* $t|t|$ on $\mathbb{R}$ and show that they are linearly independent on $\mathbb{R}$. **(2+8+4)**

4. a) Find the Legendre series of the function $f(x) = e^x$.

   b) Show that the Legendre polynomial $P_n(x)$ satisfies $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$.

   c) Define regular singular point with an example. **(6+6+2)**

5. a). Derive Bessel's function of the First kind.

   b) With usual notations for Bessel functions show that

   (i) $\frac{d}{dt}\left(t^P J_P(t)\right) = t^P J_{P-1}(t)$

   (ii) $\frac{d}{dt}\left(t^{-P} J_P(t)\right) = -t^{-P} J_{P+1}(t)$

   (9+5)

Contd...2

6 a) Let $A(t)$ be an $n \times n$ continuous matrix on I. Let $\Phi(t)$ be an $n \times n$ matrix whose i^{th} column $(i = 1,2,\ldots,n)$ is a solution of $X' = A(t)X, t \in I$. Then show that $W(t) = \det \Phi(t)$ satisfies $W'(t) = tr(A(t))\, W(t),\ t \in I$. If $t_0 \in I$ then show that $W(t) = W(t_0)\ exp \int_{t_0}^{t} tr(A(s))ds,\ t \in I$.

b) Find the fundamental matrix for $X' = A(t)X$ where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

(7+7)

7 a). Let $A(t)$ be an $n \times n$ continuous matrix on $(-\infty,\infty)$ and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $X' = A(t)X$, then show that $\Phi(t+\omega)$ is also a fundamental matrix. Show that any fundamental matrix $\Phi(t)$ can be written as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a non-singular periodic matrix of period ω and R is a constant matrix.

b) If A is an $n \times n$ constant matrix, find a fundamental matrix of $X' = A(t)X$, by considering the cases where the eigen values of A are all distinct.

(7+7)

8 a) State and prove Gronwall's inequality.

b) Find the first four Picard's approximations of the initial value problem $x' = t + x,\ x(0) = 1$. Further find the limit of these approximations.

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(7+7)

PS 564.3

Reg. No:

St Aloysius College (Autonomous)
Mangaluru
Semester III - P.G. Examination - M.Sc. Mathematics
November - 2018
COMMUTATIVE ALGEBRA

Time: 3 Hours

Max.Marks:70

PART -A

Answer any FIVE full questions.

1. a) Define a nilpotent element in a ring A. Show that the set of all nilpotent elements forms as ideal of A.

b) Define the Jacobson radical $\mathcal{J}$ of a ring A. Prove that an element $x \in \mathcal{J}$ if and only of $1 - xy$ is a unit in A, for every $y \in A$

c) Prove that in the polynomial ring $A[x]$, the nilradical ad the Jacobson radical are equal.

(5+5+4)

2 a) Prove that every non-unit in a non-zero ring A is contained in a maximal ideal.

b) Let A be a Boolean ring. Prove that every prime ideal $\mathfrak{p}$ in A is maximal and that $A/\mathfrak{p}$ is a field with two elements.

c) If $f: A \to B$ is a ring homomorphism, define the extension I^e of an ideal I of A. If I is a prime ideal of A, is I^e prime? Justify your answer.

(7+4+3)

3. a) If $\mathfrak{p}_1, \mathfrak{p}_2, .., \mathfrak{p}_n$ are prime ideals of a ring A, and I is an ideal of A, contained in $\bigcup_{j=1}^{n} \mathfrak{p}_j$, then prove that $I \subseteq \mathfrak{p}_j$ for some $j, 1 \le j \le n$.

b) If A is a ring, describe the prime spectrum of A. Further, show that spectrum A is a compact space.

(6+8)

4. a) Define a finitely generated A- module. If $0 \to M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of A-modules such that M' and M'' are finitely generated, then show that M is finitely generated.

b) Let M be a finitely generated A-module, let I be an ideal of A and φ be an A-module endomorphism of M such that $\varphi(M) \subseteq IM$. Prove that φ satisfies an equation of the form $\varphi^n + a_1\varphi^{n-1} + \cdots + a_{n-1}\varphi + a_n = 0$, where $a_i \in I, 1 \le i \le n$. Hence deduce the Nakayama's lemma.

(5+9)

5 a). Let I be an ideal of a ring A and let $S = 1 + I$. Show that S is multiplicatively closed in A. Deduce that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$.

Contd...2

b) If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A- modules, show that for a multiplicatively closed set S in A, $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact.

c) If I, J are ideals of a ring A such that J is finitely generated, show that $S^{-1}(I : J) = (S^{-1}I : S^{-1}J)$. (4+5+5)

6 a) Show that $A_{\mathfrak{p}}$ is a local ring, where $\mathfrak{p}$ is a prime ideal of A.

b) If $\varphi: M \to N$ is as A-module homomorphism, show that the following statements are equivalent:

i) φ is injective

ii) $\varphi_{\mathfrak{p}}: M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective for every prime ideal $\mathfrak{p}$ of A.

iii) $\varphi_{\mathfrak{m}}: M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m}$ of A.

c) Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists an element $s \in S$ such that $sM = 0$. (4+5+5)

7 a). Let S be a multiplicatively closed set in ring A. Prove that there exists a one-to-one correspondence between the set of all prime ideals of $S^{-1}A$ and the set of all prime ideals of A disjoint with S.

b) Let $f : A \to B$ be a ring homomorphism and let $\mathfrak{p}$ be a prime ideal of A. Prove that $\mathfrak{p}$ is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.

(8+6)

8 a) Define a minimal primary decomposition. State and prove the First Uniqueness Theorem.

b) In a Noetherian ring, prove that every irreducible ideal is primary.

(8+6)

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(199)

PH 561.3

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St Aloysius College (Autonomous)

Mangaluru

Semester III – P.G. Examination - M. Sc. Mathematics

November - 2017

COMPLEX ANALYSIS I

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following **(14x5=70)**

1. a) Define stereographic projection of the Riemann sphere onto the extended complex plane and show that circles on the Riemann sphere correspond to circles or straight lines in the extended complex plane under the stereographic projection.

   b) Find the four values of $\sqrt[4]{-1}$.

   c) Show that $\left|\frac{z_1 - z_2}{1-\bar{z}_1 z_2}\right| < 1$ if $|z_1|<1$ and $|z_2|<1$. **(10+2+2)**

2. a) If all the zeros of a polynomial $P(z)$ lie in a half-plane, then show that all the zeros of $P'(z)$ also lie in the same half-plane.

   b) Show that an analytic function in a region Ω whose derivative vanishes identically must reduce to a constant.

   c) Show that the function $f(z)=\bar{z}$ is not differentiable at any point. **(6+6+2)**

3. a) Define the exponential function e^z and show that $e^{a+b}=e^a e^b$ for all $a, b \in \mathbb{C}$.

   b) If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then show that there exists a number R, $0 \le R \le \infty$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for every z with $|z|<R$, the sum of the series is an analytic function in $|z|<R$ and it diverges for $|z|>R$. **(4+10)**

4. a) Show that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line.

   b) When do you say that the points z and z^* are symmetric with respect to the circle through z_1, z_2 and z_3?

   c) Show that $w=\frac{z-1}{z+1}$ maps the imaginary axis in the z-plane onto the circle $|w|=1$. **(9+2+3)**

Contd...2

5. a) If $p(x,y)$ and $q(x,y)$ are real or complex valued continuous functions defined in a region Ω and if γ is any curve in Ω, then show that $\int_{\gamma} pdx + qdy$ depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

b) Show that the index of 'a' with respect to a piece-wise differentiable closed curve γ which does not pass through 'a' is constant in each of the regions determined by γ. **(10+4)**

6. a) State and prove Cauchy's theorem for a rectangle.

b) If $f(z)$ is analytic on a rectangle R except at a point a in the interior of R and if $\lim_{z \to a} (z-a)f(z)=0$, then show that $\int_{\partial R} f(z)\, dz = 0$. **(10+4)**

7. a) State and prove Cauchy's integral formula.

b) Find the value of $\int_{|z|=2} \frac{e^z}{z-1} dz$.

c) Prove that every polynomial over $\mathbb{C}$ of degree >0 has atleast one root. **(7+2+5)**

8. a) State the maximum principle for analytic functions. If $f(z)$ is continuous on a closed and bounded set E and analytic in the interior of E, then show that the maximum of $|f(z)|$ on E is assumed on the boundary of E.

b) Show that a non-constant analytic function maps open sets onto open sets. **(7+7)**

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St Aloysius College (Autonomous)

Mangaluru

Semester III – P.G. Examination - M. Sc. Mathematics

November - 2017

TOPOLOGY

Time: 3 Hours | Max. Marks: 70

Answer any **FIVE FULL** questions from the following (14x5=70)

1. a) Define co-finite topology τ on X and show that (X, τ) is a topological space.

   b) If $\mathcal{B}$ and $\mathcal{B}'$ are bases for the topologies τ and τ' respectively on a set X, then show that τ' is finer than τ if and only if for each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

   c) Define the topologies of $\mathbb{R}$, $\mathbb{R}_l$ and $\mathbb{R}_k$ and compare them.

(3+5+6)

2. a) Show that $\{(x, y) \in \mathbb{R}^2 \mid x \geq 0 \text{ and } y \geq 0\}$ is closed in $\mathbb{R}^2$.

   b) If Y is a subspace of X and $A \subseteq Y$ then prove the following:

   i) Closure of A in Y is $\overline{A} \cap Y$, where $\overline{A}$ denotes closure of A in X.

   ii) A is closed in Y if and only if it equals the intersection of a closed set of X with Y.

   c) If A is a subset of topological space X then prove that $\overline{A} = A \cup A'$, where A' is the set of all limit points of A. (2+8+4)

3. a) If X is a Hausdorff space then prove that every finite set in X is closed. Is the converse true. Justify.

   b) Prove that a topological space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.

   c) Prove or disprove subspace of a Hausdorff space is Hausdorff.

(7+5+2)

4. a) If $f : X \to Y$ is continuous then prove that $f(\overline{A}) \subseteq \overline{f(A)}$, for every subset A of X.

   b) Define convergence of a sequence in a topological space and state and prove Sequence Lemma.

   c) Let X be a metrizable space. Prove that a map $f : X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X.

(2+6+6)

Contd...2

5. a) Define a connected space. Give an example each for a connected space and not a connected space.

   b) Prove that the union of a collection of connected subspaces of X that have a point in common is connected.

   c) Define connected components of X. Prove that the components of X are connected disjoint subsets of X whose union is X such that each connected subset of X intersect only one of them.

   d) If $f:[0,1]\to[0,1]$ is a continuous map then show that there exists a point $x\in[0,1]$ such that $f(x)=x$. **(3+4+5+2)**

6. If $f:X\to Y$ is a bijective continuous map where X is compact and Y is Hausdorff then prove that f is a homeomorphism. **(14)**

7. a) Define a separable space. If (X,d) is a metric space, then prove that X is separable if and only if it is second countable.

   b) Define a regular space. Prove that every compact Hausdorff space is regular. **(8+6)**

8. State and prove Tietze extension theorem. **(14)**

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PS 563.3

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St Aloysius College (Autonomous)
Mangaluru
Semester III - P.G. Examination - M.Sc. Mathematics
November - 2017
ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 Hours Max.Marks:70

PART -A

Answer any **FIVE** full questions.

1. a) Define the Wronskian of n functions. Compute the Wronskian of the functions $\sin t$, $\cos t$, $\sin 2t, t \in \mathbb{R}$.

   b) Prove that the n solutions $\phi_1, \phi_2, \dots \phi_n$, of $x^{(n)}(t) + a_1(t)\, x^{(n-1)}(t) + \dots + a_n(t)\, x(t) = 0$ on I are linearly independent on an interval I if and only if $W(\phi_1(t), \phi_2(t), \dots, \phi_n(t)) \neq 0 \quad \forall t \in I$.

   c) Use the method of undetermined coefficients to find the particular solution of $y'' - y = t^2 + 1$. **(3+8+3)**

2. a) Describe the method to find the second solution of $x'' + a_1(t)x' + a_2(t)x = 0$ on an interval I if one of its solution $\phi_1(t) \neq 0 \; \forall \, t \in I$ is given. Further show that these two solutions are linearly independent over I.

   b) Find the solution of $x'' + x = \tan t$, $0 \le t \le \frac{\pi}{2}$ using method of variation of parameters. **(8+6)**

3. a) Find the general solution of the Legendre equation $(1-x^2)y'' - 2xy' + p(p+1)y = 0$, p is a real constant.

   b) Show that the Legendre polynomial $P_n(x)$ satisfies $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$. **(8+6)**

4. a) If P_n is a Legendre polynomial, then prove that $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$.

   b) Consider the equation $2t^2x'' + t(2t+1)x' - x = 0$. Show that $t = 0$ is a regular singular point.

   c) Express $J_5(t)$ in terms of $J_0(t)$ and $J_1(t)$. **(8+2+4)**

5. a). Find the solution of the differential equation $t^2 x'' - (1+t)\, x = 0$ of the form $t^m \sum_{k=0}^{\infty} a_k t^k$, where m, a_k are constants to be determined.

   b) Find the Legendre series of the function $f(x) = x^3$.

   c) Compute the indicial polynomial and their roots for the equation $x^2y'' + \sin x\, y' + \cos x\, y = 0$. **(7+5+2)**

Contd...2

6 a) Let A be an $n \times n$ constant matrix. Find a fundamental matrix for $X' = AX$, by considering the cases when some of the eigen values of A are repeated.

b) Determine a fundamental matrix for the system $X' = AX$

where $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$

(7+7)

7 a). Let $A(t)$ be an $n \times n$ continuous matrix on I. Let $\Phi(t)$ be an $n \times n$ matrix whose i^{th} column $(i = 1, 2, \ldots, n)$ is a solution of $X' = A(t)X$, $t \in I$. Then show that $W(t) = \det \Phi(t)$ satisfies $W'(t) = tr(A(t))\, W(t)$, $t \in I$. If $t_0 \in I$ then show that $W(t) = W(t_0)\ exp \int_{t_0}^{t} tr(A(s))ds$, $t \in I$.

b) Let $A(t)$ be an $n \times n$ continuous matrix on $(-\infty, \infty)$ and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $X' = A(t)X$, then show that $\Phi(t + \omega)$ is also a fundamental matrix. Show that any fundamental matrix $\Phi(t)$ can be written as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a non-singular periodic matrix of period ω and R is a constant matrix.

(7+7)

8 a) Let $f(t, x)$ be a continuous function defined over a rectangle $R = \{(t, x) : |t - t_0| \le p, |x - x_0| \le q\}$ where p, q are some positive real numbers. Let $\frac{\partial f}{\partial x}$ be defined and continuous on R. Then prove that $f(t, x)$ satisfies the Lipschitz condition on R.

b) Let f, g, h be continuous non-negative functions for $t \ge t_0$.
If $f(t) \le h(t) + \int_{t_0}^{t} f(s)g(s)\, ds$, $t \ge t_0$ then prove that

$f(t) \le h(t) + \int_{t_0}^{t} h(s)g(s) \exp[\int_{s}^{t} g(u)du]\, ds$, $t \ge t_0$.

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(7+7)

PS 564.3

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St Aloysius College (Autonomous)

Mangaluru

Semester III - P.G. Examination - M.Sc. Mathematics

November - 2017

COMMUTATIVE ALGEBRA

Time: 3 Hours

Max.Marks:70

PART -A

Answer any FIVE full questions.

1. a) Define the nilradical of a ring A. Show that the nilradical is the intersection of all prime ideals of A.

b) Prove that in a polynomial ring A[x], a polynomial $f(x) = a_0+a_1x+\cdots+a_nx^n$ is a nilpotent element if and only if a_i is nilpotent in A, for each $i, 0 \le i \le n$.

(10+4)

2 a) Define a local ring. Prove that a local ring has no non-trivial idempotent element.

b) Let A be a ring in which every element x satisfies $x^n = x$, for some integer $n \ge 2$. Show that every prime ideal in A is maximal.

c) Define the radical $r(I)$ of an ideal I of a ring A. If I, J are ideals of A. Prove that I, J are coprime if and only if $r(I)$ and $r(J)$ are coprime.

(4+5+5)

3. If I_j, $1 \le j \le n$ are ideals of a ring A such that $I_j + I_k = A$, for all $j \ne k, 1 \le j, k \le n$, prove that $A/\prod_{j=1}^{n} I_j \cong \prod_{j=1}^{n}\left(A/I_j\right)$ (14)

4 a) Construct the spectrum of a ring A. show that spec (A) of a Boolean ring is a Hausdorff space.

b) If $I_1, I_2, \dots, I_n$ are ideals of a ring A and $\mathfrak{p}$ is a prime ideal of A containing $\bigcap_{j=1}^{n} I_j$, show that $\mathfrak{p} \supseteq I_j$ for some j, $1 \le j \le n$.

(9+5)

5 a). State and prove Nakayama's lemma.

b) If $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ is an exact sequence of A-modules, then prove that for any A- module N, the sequence $0 \to Hom\,(M'',N) \xrightarrow{\bar{v}} Hom(M,N) \xrightarrow{\bar{u}} Hom\,(M',N)$ is exact, where $\bar{v}(f) = f \circ v$ for every $f \in Hom\,(M'',N)$, and $\bar{u}(g) = g \circ u,\ \forall g \in Hom\,(M,N)$.

(5+9)

Contd...2

6 a) Let $g: A \to B$ be a ring homorphism and S be a multiplicatively closed set in A such that $g(s)$ is a unit in B for every $s \in S$. Prove that there is a unique ring homomorphism $h: S^{-1}A \to B$ such that $g = h \circ f$, where $f: A \to S^{-1}A$ is given by $f(a) = \frac{a}{1}$ $\forall\, a \in A$.

b) Show that every ideal in $S^{-1}A$ is an extended ideal.

c) Let M be an A-module. Prove that the following statements are equivalent:

i) $M = 0$

ii) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ of A.

iii) $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ of A.

(6+3+5)

7 a). Let A be a ring and S be a multiplicatively closed set in A. Prove that there is one-to-one correspondence between the set of all prime ideals of $S^{-1}A$ and the set of all prime ideals of A which do not meet S.

b) If M is a finitely generated A- module and S is a multiplicatively closed set in A, show that $S^{-1}(Ann(M)) = Ann(S^{-1}M)$. (9+5)

8 a) Define a primary ideal in a ring A. Show that the intersection of finitely many $\mathfrak{p}$-primary ideals is $\mathfrak{p}$-primary.

b) State and prove the Second Uniqueness Theorem for primary decomposition. (4+10)

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