PH 561.3

Reg. No: | | | | | | |

St Aloysius College (Autonomous)

Mangaluru

Semester III - P.G. Examination - M. Sc. Mathematics

JANUARY-2021

COMPLEX ANALYSIS I

Time: 3 Hours | Max. Marks: 70

Answer any FIVE FULL questions from the following: (14x5=70)

1. a) If $a, b \in \mathbb{C}$ with $|a| < 1$ and $|b| < 1$, then prove that $\left|\frac{a-b}{1-\bar{a}b}\right| < 1$.

   b) In the spherical representation of the extended complex plane, show that the circles and straight lines in the extended complex plane correspond to the circles on the Riemann sphere.

   c) What does $E = \{z \in \mathbb{C}: |z+i| = 2|z|\}$ represent in the complex plane?

(2+10+2)

2. a) If all the zeros of a polynomial $P(z)$ lie in a half-plane, then show that all the zeros of $P'(z)$ also lie in the same half-plane.

   b) State and prove a necessary and sufficient condition for a function $f(z) = u(z) + iv(z)$ to be analytic in the region Ω.

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

(5+9)

3. a) If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then prove that there exists a number R, $0 \le R \le \infty$ such that $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for every z with $|z| < R$, the sum of the series is an analytic function in $|z| < R$ and it diverges for $|z| > R$.

   b) Show that $e^{a+b} = e^a . e^b$, for all $a, b \in \mathbb{C}$. Deduce that e^z has no zeros in the complex plane.

(10+4)

4. a) Show that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line.

   b) If a linear transformation carries a circle C_1 into a circle C_2, then prove that it transforms any pair of symmetric points w.r.to C_1 into a pair of symmetric points w.r.to C_2.

(9+5)

Contd...2

5. a) Find the linear transformation which maps $1, 0, i$ of the z-plane onto $-i, -1, 0$ of the w-plane respectively.

   b) If $p(x, y)$ and $q(x, y)$ are real or complex valued continuous functions defined on a region Ω and if γ is any curve in Ω, then show that $\int_{\gamma} p dx + q dy$ depends only on the end points of γ if and only if there exists a function $U(x, y)$ in Ω with $\frac{\partial U}{\partial x} = p$ and $\frac{\partial U}{\partial y} = q$.

   c) Let $f(z) = \frac{z-i}{z}$ and $(t) = 2e^{it}$, $0 \le t \le 2\pi$. Evaluate $\int_{\gamma} f(z)\, dz$.

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

(3+9+2)

6. a) Show that the index of 'a' with respect to a piece-wise differentiable closed curve γ which does not pass through 'a' is constant in each of the regions determined by γ.

   b) State and prove Cauchy's theorem for a rectangle.

(4+10)

7. a) State and prove Cauchy's integral formula.

   b) Find the value of $\int_{|z|=2} \frac{e^z}{z-1} dz$.

   c) Prove that every polynomial over $\mathbb{C}$ of positive degree has atleast one root.

(7+2+5)

8. a) State the maximum principle for analytic functions. If $f(z)$ is continuous on a closed and bounded set E and analytic in the interior of E, then show that the maximum of $|f(z)|$ on E is assumed on the boundary of E.

   b) Show that a non-constant analytic function maps open sets onto open sets. **(7+7)**

****

PH 562.3 Reg No.

St Aloysius College (Autonomous)

Mangaluru

Semester III - P.G. Examination –M.Sc. Mathematics

JANUARY-2021

TOPOLOGY

Time : 3 hours Max. Marks :70

Answer any FIVE FULL questions from the following:

1. a) Define a basis $\mathcal{B}$ for a topology on a set X and the topology τ generated by $\mathcal{B}$ on X. Prove that τ equals the collection of all unions of elements of $\mathcal{B}$.

   b) If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then prove that the collection $\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$ is a basis for the topology of $X \times Y$.

   c) Prove that a subset A of a topological space X is closed if and only if A contains all its limit points.

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

(5+6+3)

2. a) Define relative topology and illustrate with an example. If (X, τ) is any topological space and $Y \subset X$ then prove that the family $\tau_y = \{Y \cap G : G \in \tau\}$ forms a topology on Y.

   b) Let X be an ordered set in the order topology. Let Y be a subset of X that is convex in X. Then prove that the order topology on Y is the same as the topology Y inherits as a subspace of X.

   c) Let X be a topological space and $A \subseteq X$. Define the interior of A and the boundary of A. Prove that closure of A is the disjoint union of $Int(A)$ and $Bd(A)$.

(4+5+5)

3. a) Define open maps and closed maps. Show that a continuous open map need not be closed.

   b) State and prove sequence lemma.

   c) Let X be a metrizable space. Prove that a map $f: X \to Y$ is continuous if and only if $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X. Also Prove the converse holds if X is metrizable.

5+5+4)

4. a) Prove that the real line $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.

   b) Prove that every path connected space is connected but not conversely.

   c) Prove that a space X is locally connected if and only if for every open set U of X, each component of U is open in X.

(5+4+5)

Contd...2

5. a) Prove that every compact subspace of a Hausdorff space is closed.

   b) Define a homeomorphism. If $f: X \to Y$ is a bijective continuous map, where X is compact and Y is Haursdorff, then show that f is a homeomorphism.

(7+7)

6. a) Define a locally compact space. Prove that every compact space is locally compact. Give an example to illustrate that a locally compact space need not be compact.

   b) Show that product of two Hausdorff spaces is again Hausdorff.

   c) Define a second countable space. If X is second countable, show that every open cover of X contains a countable subcollection covering X.

(3+4+7)

7. a) Define a separable space. Prove that every second countable space is separable. Show that the converse holds if X is metrizable.

   b) Define a regular space. Show that every locally compact Hausdorff space is regular.

(7+7)

8. State and prove Tietze extension theorem. (14)

****

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

PS 563.3 Reg. No:

St Aloysius College (Autonomous)

Mangaluru

Semester III - P.G. Examination - M.Sc. Mathematics

JANUARY-2021

ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 Hours Max.Marks:70

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

Answer any FIVE full questions. (14x5=70)

1. a) If $\phi_1(t), \phi_2(t), \phi_3(t), \ldots\ldots, \phi_n(t)$ are solutions of the equation $x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots\ldots+a_n(t)x(t)=0$, then prove that they are linearly independent on the interval I if and only if $W(\phi_1(t), \phi_2(t), \ldots\ldots, \phi_n(t)) \neq 0 \quad \forall t \in I.$

   b) Compute the Wronskian of the two independent solutions of $x''-2tx'+2x=0$ for $t>0$ given that $x_1(t)=t$ is a solution of the equation.

   c) Show that the function t^2 and $t|t|$ are not linearly independent on $[-1,0]$.

   (7+4+3)

2. a) Describe the method of reduction of order to find the solution of $x''+a_1(t)x'+a_2(t)x=0$.

   b) Describe the method of variation of parameters to find the solution of $x''+a_1(t)x'+a_2(t)x=b(t)$. (7+7)

3. a) State and prove formula for the Wronskian.

   b) Solve $x''-2x'+x=t+e^t$ using the method of undetermined coefficients.

   c) Show that the particular solution of the initial value problem $x''+x=g(t)$ with $x(0)=0,\ x'(0)=0$ is $x(t)=\int_0^t \sin(t-s)g(s)\,ds$. (7+3+4)

4. a) State and prove orthogonal property of Legendre polynomials $P_n(t)$.

   b) Find the Legendre series of the function $f(x)=x^2$.

   c) Prove that $(n+1)P_{n+1}(t)=(2n+1)tP_n(t)-nP_{n-1}(t)$, where $P_n(t)$ is Legendre polynomial. (7+3+4)

5. a) Obtain the series solutions of the Bessel equation $t^2x''+tx'+(t^2-n^2)x=0$.

   b) Prove that $\left(J_{\frac{1}{2}}(t)\right)^2+\left(J_{-\frac{1}{2}}(t)\right)^2=\frac{2}{\pi t},\ t>0$. (8+6)

6. a) Let $A(t)$ be an $n\times n$ continuous matrix on an interval I. Let $\Phi(t)$ be an $n\times n$ matrix whose i^{th} column, $i=1,2,\ldots.n$ is a solution of $x'=A(t)x$, $t\in I$. Then show that $w(t)=\det\Phi(t)$ satisfies $w'(t)=tr(A(t))w(t)$, $t\in I$. If $t_0\in I$ then show that $w(t)=w(t_0)\,exp\left(\int_{t_0}^t tr(A(s))\,ds\right)$, $t\in I$.

b) Using the method of successive approximation, solve the IVP $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} t & 0 \\ 0 & 2t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(8+6)

7. a) Let $A(t)$ be an $n \times n$ continuous matrix on $(-\infty, \infty)$ and be periodic with period ω. If $\phi(t)$ is a fundamental matrix for the system $x' = A(t)\,X$, then show that $\phi(t+\omega)$ is also a fundamental matrix. Show that any fundamental matrix $\phi(t)$ can be written as $\phi(t) = P(t)e^{tR}$, where $P(t)$ is a non-singular periodic matrix of period ω and R is a constant matrix.

b) Find the fundamental matrix of $x' = A(t)x$, where $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$.

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

(8+6)

8. a) Prove that a function ϕ is a solution of the IVP $y' = f(x,y)$, $y(x_0) = y_0$ on an interval I if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t,y)dt$.

b) Find the first four approximations of the initial value problem $x'(t) = 1 + tx$, $x(0) = 1$.

(8+6)

****

PS 564.3

Reg. No. : | | | | | | |

St Aloysius College (Autonomous)
Mangaluru
Semester III– P.G. Examination – M.Sc. Mathematics
JANUARY-2021
COMMUTATIVE ALGEBRA

Time: 3 hrs. **Max Marks: 70**

PART - A

Answer any FIVE FULL questions from the following: **(14x5=70)**

1. a) Prove that the nilradical of a ring A is the intersection of all prime ideals of A.

   b) Define the radical $r(I)$ of an ideal I of a ring A.
   Prove that $r(I+J) = r(r(I)+r(J))$. Hence prove that $r(I)$ and $r(J)$ are coprime if and only if I and J are coprime. **(9+5)**

ST.ALOYSIUS COLLEGE
PG Library
MANGALORE-575 003

2. If $I_1, I_2, \dots, I_n$ are ideals in a ring A with $I_j + I_k = (1)$ whenever $j \neq k$, $1 \leq j,k \leq n$, then show that $A/\prod_{j=1}^{n} I_j \cong \prod_{j=1}^{n} A/I_j$ **(14)**

3. a) If $P_1, P_2, \dots, P_n$ are prime ideals of a ring A such that an ideal I of A is contained in $\bigcup_{j=1}^{n} P_j$, then prove that $I \subseteq P_k$ for some k, $1 \leq k \leq n$.

   b) Define the Jacobson radical J of a ring A. Show that $x \in J$ if and only if $1 - xy$ is a unit in A, for every $y \in A$. **(8+6)**

4. a) Let M be a finitely generated A – module and let I be an ideal of A contained in the Jacobson radical of A. Then prove that $IM = M$ implies $M = 0$.

   b) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A – modules. If M' and M'' are finitely generated, then show that M is also finitely generated. **(6+8)**

5. a) If $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of A-modules and S is a multiplicatively closed subset of a ring A, show that $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ is exact.

   b) If S is a multiplicatively closed subset of a ring A, prove that the map $P \mapsto S^{-1}P$ gives a bijective correspondence between the set of all prime ideals of A which do not meet S and the set of all prime ideals of $S^{-1}A$. **(5+9)**

6. a) If P is a prime ideal of a ring A, show that A_p is a local ring.

   b) If $f: A \to B$ is a ring homomorphism and P is a prime ideal of A, prove that P is the contraction of a prime ideal of B if and only if $P^{ec} = P$. **(5+9)**

7. a) State and prove the first uniqueness theorem for primary decomposition.

   b) Obtain a minimal primary decomposition of the ideal $I = (x^2, xy)$ in $A = K[x, y]$, where K is a field. **(10+4)**

8. Prove that a ring A is noetherian if and only if every prime ideal of A is finitely generated. **(14)**

****