Reg. No. | | | | | | |

PII 561.3

St. Aloysius College (Autonomous)

Semester III – P.G. Examination – M. Sc. Mathematics

February - 2022

Complex Analysis I

Time : 3 Hours Max. Marks : 70

Answer any FIVE FULL questions from the following (14 × 5 = 70)

1. (a) In the spherical representation of the extended complex plane, show that the circles and straight lines in the extended complex plane correspond to the circles on the Riemann sphere.
   (b) Find the distance between the stereographic projections of $1 + i\sqrt{3}$ and $1 - i\sqrt{3}$.
   (c) Find the values of $\sqrt{1+i}$. (9+3+2)

2. (a) State and prove a necessary and sufficient condition for a function $f(z) = u(z) + i\,v(z)$ to be analytic in a region Ω.
   (b) State and prove the Lucas's theorem. (9+5)

3. (a) Let $\sum_{n=1}^{\infty} f_n$ be a series of complex-valued functions where each f_n is defined on a subset E of $\mathbb{C}$. If $\sum_{n=1}^{\infty} a_n$ is a convergent majorant of $\sum_{n=1}^{\infty} f_n$ on E, then show that $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.
   (b) If $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$ is a power series, then show that there exists a real number R with $0 \le R \le \infty$, called the radius of convergence, such that in $|z| < R$, the sum of the series is an analytic function, the derivative can be obtained by term-wise differentiation, and the derived series has the same radius of convergence. (5+9)

4. (a) State and prove the Cauchy's necessary and sufficient condition for uniform convergence of a sequence.
   (b) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.
   (c) Show that e^{iz} has the least positive period 2π and all other periods are integer multiplesof 2π.
   (5+2+7)

5. (a) Show that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie on a circle or a straight line.

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(b) If z_1, z_2, z_3, z_4 are four distinct points of the extended complex plane and T is a linear transformation, then show that $(Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$.

(c) Show that any linear transformation which maps z_2, z_3, z_4 to $1, 0, \infty$, respectively, is unique.

(7+4+3)

6. (a) Let Ω be a region and γ be any piecewise differentiable arc in Ω. Show that the integral $\int_\gamma f\,dz$, with continuous f, depends only on the end points of γ if and only if f is the derivative of an analytic function in Ω.

(b) Compute $\displaystyle\int_{|z|=2} \frac{dz}{z^2+1}$.

(10+4)

7. (a) Let the function $f(z)$ be analytic on a set R' obtained from a rectangle R by omitting a finite number of interior points $\zeta_1, \zeta_2, \ldots, \zeta_n$. If $\lim_{z\to\zeta_i} (z-\zeta_i)f(z) = 0$, for each i, $1 \le i \le n$, then show that $\int_{\partial R} f(z)\,dz = 0$.

(b) State and prove the Cauchy's theorem for a disk.

(5+9)

8. (a) Show that the zeros of an analytic function which does not vanish identically are isolated.

(b) State and prove the open mapping theorem.

(c) State the maximum principle for analytic functions. If $f(z)$ is continuous on a closed and bounded set E and analytic on the interior of E, then show that the maximum of $|f(z)|$ on E is assumed on the boundary of E.

(3+4+7)

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St Aloysius College (Autonomous)

Mangaluru

Semester III - P.G. Examination - M.Sc. Mathematics

February - 2022

ORDINARY DIFFERENTIAL EQUATIONS

Time: 3 hrs. **Max Marks: 70**

Note: Answer any five full questions. **(14x5=70)**

1. a) If $\phi_1(t), \phi_2(t), \phi_3(t), \ldots\ldots, \phi_n(t)$ are solutions of the equation $x^{(n)}(t)+a_1(t)x^{(n-1)}(t)+\ldots\ldots+a_n(t)x(t)=0$, then prove that they are linearly independent on I if and only if $W(\phi_1(t), \phi_2(t), \ldots\ldots, \phi_n(t)) \neq 0 \quad \forall t \in I$.

   b) State and prove Abel's formula for n^{th} order linear homogeneous differential equation. **(8+6)**

2. a) Let $\phi_1, \phi_2, \phi_3, \ldots\ldots, \phi_n$ be n linearly independent solutions of the equation, $L_n(x) \equiv x^{(n)}(t)+b_1(t)x^{(n-1)}(t)+\ldots\ldots+b_n(t)x(t)=0$ existing on I. Let the real or complex valued function h be defined and continuous on I. Further assume that $W(t)=W(\phi_1(t), \phi_2(t), \cdots \phi_n(t))$ and $W_k(t)$ denote $W(t)$ with k^{th} column replaced by n elements $0, 0, 0, \cdots, 1$. Then prove that a particular solution $x_p(t)$ of $L_n(x)=h(t)$ is given by $x_p(t)=\sum_{k=1}^{n}\phi_k(t)\int_{t_0}^{t}\frac{W_k(s)h(s)}{W(s)}ds$; $t, t_0 \in I$.

   b) Find a general solution of $L_2(x)=x''-2tx'+2x=0$. **(8+6)**

3. a) Prove that the functions x^2 and $|x|x$ are linearly independent on $[-1,-1]$ but they are linearly dependent on $[-1,0]$ and $[0,1]$.

   b) Solve $x^{(4)}+4x=2\sin t+1+3t^2+4e^t$. **(6+8)**

4. a) State and prove orthogonal property of Legendre polynomials.

   b) Find the Legendre series of the function $f(x)=e^x$. **(8+6)**

5. a) Derive Bessel's function of the first kind.

   b) With usual notations for Bessel functions show that:

   i) $\frac{d}{dt}\left(t^p J_p(t)\right)=t^p J_{p-1}(t)$

   ii) $\frac{d}{dt}\left(t^{-p} J_p(t)\right)=-t^{-p} J_{p+1}(t)$. **(9+5)**

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Semester III - P.G. Examination - M.Sc. Mathematics

February - 2022

TOPOLOGY

Time: 3 hrs. **Max Marks: 70**

Answer any FIVE FULL questions from the following: **(14x5=70)**

1. a) Define a closed set in a topological space. Prove that the union of two closed sets is closed.
   b) Define the topologies of $\mathbb{R}$, $\mathbb{R}_l$ *and* $\mathbb{R}_k$ and compare them.
   c) Let Y be a subspace of topological space X. Prove that a set A is closed in Y if and only if $A = C \cap Y$ for some closed set C in X.

   (2+8+4)

2. Let X be a topological space and $A \subseteq X$. Define closure of A in X. Prove the following :
   a) $x \in \bar{A}$ if and only if every neighbourhood of x intersects A.
   b) $\bar{A} = A \cup A'$ where A' denotes the set of all limit points of A.
   c) A is a closed subset of X if and only if $A' \subseteq A$.

   (2+5+4+3)

3. a) If X is topological space then prove that X is T_1 if and only if every finite subset of X is closed.
   b) Define a Housdroff space. Prove that a topological space X is Housdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X : x \in X\}$ is closed in $X \times X$.
   c) Prove that every metric space is Housdorff.

   (5+6+3)

4. a) Let X and Y be topological spaces and $f: X \to Y$ be a map. Show that the following are equivalent:
      i) f is continuous
      ii) For every subset A of X, $f(\bar{A}) \subseteq \overline{f(A)}$
      iii) For every closed subset B of Y the set $f^{-1}(B)$ is closed in X.
      iv) For each x in X and each neighbourhood V of $f(x)$, there is a neighbourhood $\mathcal{U}$ of x, such that $f(\mathcal{U}) \subseteq V$.
   b) State and prove sequence lemma.

   (9+5)

5. a) Define a connected space. Prove that the union of collection of connected subspaces of X that have a point in common is connected.
   b) Prove that the product of two path connected spaces is path connected.
   c) Show that a path connected space is always connected.

   (6+6+2)

6. If $f: X \to Y$ is a continuous bijective map from a compact space X into a Housdroff space Y then prove that f is a homeomorphism.

   (14)

7. a) Define a second countable space. Prove that a metric space (X, d) is separable if and only if it is second countable.
   b) Prove that every compact Housdroff space is regular.

   (8+6)

8. State and prove Urysohn's lemma

   (14)

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6. a) Let $A(t)$ be an $n \times n$ continuous matrix valued function on I. Let $\Phi(t)$ be a fundamental matrix of the system $x' = A(t)x$, $t \in I$. Then show that $\det(\Phi(t))' = tr(A(t))\det(\Phi(t))$, $t \in I$.

b) Find the fundamental matrix for $x' = Ax$ where $A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 3 & 0 \end{bmatrix}$ **(7+7)**

7. a) Let $A(t)$ be an $n \times n$ continuous matrix on I and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system, $X' = A(t)X$, then show that $\Phi(t+\omega)$ is also a fundamental matrix. Show that any fundamental matrix $\Phi(t)$ can be written as $\Phi(t) = P(t)e^{tR}$, where $P(t)$ is a non-singular matrix of period ω and R is a constant matrix.

b) Show that the set of all solutions of the system $X'(t) = A(t)X(t)$, $t \in I$ forms an n-dimensional vector space over the field of complex numbers. **(7+7)**

8. State and prove Picard's theorem. **(14)**

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St. Aloysius College (Autonomous), Mangaluru

Semester III P. G. Examination - M. Sc. Mathematics

February 2022

Commutative Algebra

Time : 3 Hours | Max. Marks : 70

Answer any FIVE full questions.

1. (a) Define the nilradical of a ring. Prove that the nilradical of A is the intersection of all prime ideals of A.

   (b) When do you say that two ideal I and J in a ring are coprime? Show that two ideals in A are coprime if and only if their radicals are coprime in A. **(8+6)**

2. (a) For any prime ideal P in a ring A, and $n \in \mathbb{N}$, show that the radical of P^n is P.

   (b) If U denotes the class of all units in A, then show that the Jacobson radical $J(A) = \{x \in A \ : \ 1 - xy \in U, \ \forall \ y \in A\}$.

   (c) If I and J are coprime ideals in a ring A, show that $A/IJ \cong A/I \times A/J$. **(4+4+6)**

3. (a) Prove or disprove: Exension of a prime ideal is a prime ideal.

   (b) Let $I_1, I_2, \ldots, I_n$ be ideals in a ring A. If $\bigcap_{i=1}^{n} I_i$ is a prime ideal P in A, then show that $P = I_j$ for some j.

   (c) Define the prime spectrum Spec(A) of a ring A. Prove that Spec(A) is a compact topological space. **(2+6+6)**

4. (a) Prove that a nonzero A-module M is isomorphic to a quotient of A^n for some $n \in \mathbb{N}$ if and only if M is a finitely generated A-module.

   (b) State and prove the Nakayama's lemma.

   (c) If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A modules such that M' and M'' are finitely generated, then show that M is also a finitely generated A-module. **(5+5+4)**

contd...2

5. (a) Show that the operation S^{-1} is exact.

(b) Let M be an A-module. Prove that the following statements are equivalent:

(i) $M = 0$,

(ii) $M_{\mathcal{P}} = 0$ for all prime ideals $\mathcal{P}$ of A,

(iii) $M_{\mathcal{M}} = 0$ for all maximal ideals $\mathcal{M}$ of A.

(c) Show that the operation S^{-1} commutes with formation of finite sums, intersections and radicals. (4+4+6)

6. (a) Let $f : A \to B$ be a ring homomorphism and let P be a prime ideal of A. Prove that P is a contraction of a prime ideal of B if $P^{ec} = P$.

(b) Let S be a multiplicatively closed subset of a ring A, and let M be a finitely generated A-module. Prove that $S^{-1}M = 0$ if and only if there exists $s \in S$ such that $sM = 0$.

(c) If I is an ideal of a ring A, then show that $S = 1 + I$ is a multiplicatively cosed subset of A. Further deduce that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}A$. (6+4+4)

7. (a) Define a primary ideal of a ring A. Is every primary ideal prime? Justify your answer.

(b) Let S be a multiplicatively closed subset of a ring A, and let Q be a P-primary ideal. Prove the following:

(i) If $S \cap P \neq \emptyset$, then $S^{-1}Q = S^{-1}A$.

(ii) If $S \cap P = \emptyset$, then $S^{-1}Q$ is $S^{-1}P$-primary.

(c) State and prove the second uniqueness theorem for primary decomposition. (2+4+8)

8. (a) Give an example for an A-module which satisfies d.c.c. but not a.c.c.

(b) Prove that a ring A is Noetherian if and only if the polynomial ring $A[x]$ is Noetherian. (2+12)

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