PH 561.4 Reg. No:

St Aloysius College (Autonomous)

Mangaluru

Semester IV – P.G. Examination - M. Sc. Mathematics

September - 2020

MEASURE THEORY AND INTEGRATION

Time: 3 Hours **Max. Marks: 70**

Answer any <u>FIVE FULL</u> questions from the following **(14x5=70)**

1. a) Define the Lebesgue outer measure $m^*(A)$ of any subset A of $\mathbb{R}$. Show that m^* is monotone and translation invariant.

   b) Prove that outer measure of an interval equals its length. **(5+9)**

2. a) For any sequence of sets $\{E_i\}$, show that $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$. Hence show that every countable set has measure zero.

   b) Let $\{E_i\}$ be a sequence of measurable sets. Prove the following:

   i) if $E_1 \subseteq E_2 \subseteq \ldots$, then $m(\lim E_i) = \lim m(E_i)$

   ii) if $E_1 \supseteq E_2 \supseteq \ldots$, and $m(E_1) < \infty$, then $m(\lim E_n) = \lim m(E_n)$.

   (4+10)

3. a) Let c be any real number and let f and g be real valued measurable functions defined on the same measurable set E. Prove that cf, $f+g$, fg are measureable.

   b) If $\{f_n\}$ is a sequence of measurable functions defined on the same measurable set, then show that $\sup f_n$ is measurable.

   c) If f is a measurable function, then show that $|f|$ is measurable. Is the converse true? Justify? **(6+4+4)**

4. a) Show that if f is a non-negative measurable function, then show that $f = 0$ *a.e.* if and only if, $\int f\,dx = 0$.

   b) State and prove Fatou's lemma. **(4+10)**

5. a) Let $\{f_n\}$ be a sequence of integrable functions such that $\sum_{n=1}^{\infty} \int |f_n|\,dx < \infty$. Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges *a.e.*, its sum $f(x)$ is integrable and $\int f\,dx = \sum_{n=1}^{\infty} \int f_n\,dx$.

   b) If f is Riemann integrable and bounded over the finite interval $[a,b]$ then prove that f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$. **(6+8)**

6. a) Let f and g be integrable functions. Prove the following:

   i) af is integrable and $\int af\,dx = a\int f\,dx$, $a \in \mathbb{R}$.

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ii) $f+g$ is integrable and $\int (f+g\,dx) = \int f\,dx + \int g\,dx$.

iii) If $f \leq g$ $a.e.$ then $\int f\,dx \leq \int g\,dx$.

iv) If A and B are disjoint measurable sets, then $\int_A f\,dx + \int_B f\,dx = \int_{A\cup B} f\,dx$.

b) If f is measurable, $m(E) < +\infty$ and $A \leq f \leq B$ on E, then prove that $A\,m(E) \leq \int_E f\,dx \leq B\,m(E)$ on E. **(10+4)**

7. a) State and prove Jensen's inequality. When does equality occurs? Discuss. **(9+5)**

b) Prove Minkowski's inequality.

8. a) Define signed measure on a measure space. **(2+12)**

b) State and prove Jordan decomposition theorem.

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PH 562.4 **Reg. No:**

St. Aloysius College (Autonomous)

Mangalore

Semester IV - P.G. Examination - M. Sc. Mathematics

September - 2020

COMPLEX ANALYSIS - II

Time: 3 Hours **Max.Marks:70**

Answer any FIVE FULL questions from the following: **(14×5=70)**

1. a) Prove that the region obtained from a simply connected region by removing m points has the connectivity $m+1$.

   b) Prove that a region Ω is simply connected if and only if $\eta(\gamma, a)=0$ for all cycles 'γ' in Ω and all points 'a' which do not belong to Ω. **(2+12)**

2. a) Find a homology basis for the annulus defined by $r_1<|z|<r_2$.

   b) State and prove the Residue theorem.

   c) Evaluate $\int_C \frac{\sin z}{\left(z-\frac{\pi}{4}\right)^3}\,dz$, where C is the circle $\left|z-\frac{\pi}{4}\right|=\frac{1}{2}$. **(4+8+2)**

3. a) State and prove the Rouche's theorem.

   b) Evaluate $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta$

   c) Evaluate $\int_0^{\infty} \frac{dx}{\left(1+x^2\right)^2}$. **(5+5+4)**

4. a) State and prove the mean value property for harmonic functions.

   b) Let u be a bounded harmonic function in $0<|z|<\rho$. Show that the origin is a removable singularity in the sense that u can be extended to a harmonic function in $|z|<\rho$, when $u(0)$ is properly defined.

   c) Suppose that $f(z)$ is analytic in $|z|\le 1$ and $f(z)\in \mathbb{R}$ if $|z|=1$. Show that $f(z)$ is a constant function. **(7+5+2)**

5. State and prove the Poisson's formula. **(14)**

6. a) Suppose that $f_n(z)$ is analytic in the region Ω_n, and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region Ω uniformly on every compact subset of Ω, then prove that $f(z)$ is analytic in Ω.

   b) State and prove the Hurwitz theorem. **(7+7)**

Contd...2

7. a) If $f(z)$ is analytic in a region Ω containing a, then show that the representation $f(z)=f(a)+\frac{f'(a)}{1!}(z-a)+....+\frac{f^{(n)}(a)}{n!}(z-a)^n+....$ is valid in the largest open disk of center a contained in Ω.

b) Find the Taylor's series for $f(z)=\frac{1}{z}$ about $z=1$.

c) Expand $f(z)=\frac{z}{(z-1)(2-z)}$ in a Laurent's series valid for $|z-1|>1$. **(8+2+4)**

8. a) Show that $\frac{\pi^2}{\sin^2 \pi z}=\sum_{n=-\infty}^{\infty}\frac{1}{(z-n)^2}$.

b) Prove that the necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty}|a_n|$.

c) Prove that $f(z)$ is an entire function without zeros if and only if $f(z)=e^{g(z)}$, where $g(z)$ is an entire function. **(6+5+3)**

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St. Aloysius College (Autonomous)
Mangaluru
Semester IV- P.G. Examination - M.Sc. Mathematics
September - 2020

FUNCTIONAL ANALYSIS

Time: 3 Hours **Max.Marks:70**

Answer any FIVE Full questions. **(14X5=70)**

1. a) Let X be a complete metric space and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n)\to 0$. Then prove that $\bigcap_{n=1}^{\infty} F_n$ contains exactly one point. Show that $\bigcap_{n=1}^{\infty} F_n$ may be empty if each 'F_n' in the hypothesis is replaced by a 'nonempty open set'.

   b) State and prove Baire's category theorem. **(8+6)**

2. Let M be a closed linear subspace of a normed linear space N. Then prove that the quotient space N/M forms a normed linear space $w.r.t$ the norm given by $\| x+M \| = \inf\{\| x+M \| : m\in M\}$. Further, show that if N is a Banach space, then so is N/M. **(14)**

3. a) Let N, N' normed linear spaces and T be a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent to one another.
      i. T is continuous
      ii. T is continuous at the origin.
      iii. There exists a real number $K\geq 0$ with the property that $\| T(x) \|\leq K\| x \|$, for every $x\in N$.
      iv. If $S=\{x\in N : \| x \| \leq 1\}$ is the closed unit sphere in N, then its image $T(S)$ is a bounded set in N'.

   b) Let N be a finite dimensional normed linear space with dimension $n > o$ and let $(v_1, v_2, \ldots, v_n)$ be a basis for N. If $T: N\to l^n$ is given by $T(x) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, whenever $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$, then show that T is continuous. **(7+7)**

4. a) Prove that every closed and bounded subset of a finite dimensional normed linear space is compact.

   b) If N is normed linear space prove that there exists an isometric isomorphism of N into N^{**}.

   c) If N is a normed linear space and x_0 is a nonzero vector in N, then show that there exists a functional f_0 in N^* such that $f_0(x_0) = \| x_0 \|$ and $\| f_0 \| = 1$. **(4+7+3)**

Contd...2

5. State and prove the open mapping theorem. (14)

6. a) Show that the parallelogram law is not true in $l_1^n (n > 1)$.

b) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a nonzero vector z_0 in H such that $z_0 \perp M$.

c) If M is a proper closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$. (2+5+7)

7. a) Let H be a Hilbert space and let f be an arbitrary functional in H^*. Then prove that there exists a unique vector y in H, such that $f(x) = < x, y >$, for every x in H.

b) If T is an operator on a Hilbert space H such that $\langle Tx, x \rangle = 0$ for all $x \in H$, then show that $T = 0$.

c) If T is an operator on a Hilbert space H then show that the following are equivalent:

i) $T^*T = I$

ii) $\langle Tx, Ty \rangle = \langle x, y \rangle$, for all $x, y \in H$

iii) $\|Tx\| = \|x\|$ for all $x \in H$.

(7+3+4)

8. State and prove that spectral theorem for a finite dimensional Hilbert space. (14)

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St Aloysius College (Autonomous)

Mangaluru

Semester IV – P.G. Examination – M.Sc. MATHEMATICS

September - 2020

PARTIAL DIFFERENTIAL EQUATIONS

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following: **(14x5=70)**

1.a) Show that the general solution of the linear partial differential equation $P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z)$ is $F(\phi(x,y,z), \psi(x,y,z) = 0$, where F is an arbitrary function and $\phi(x,y,z) = C_1$ and $\psi(x,y,z) = C_2$ form a solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

b) Verify that the equation $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$ is integrable and find its primitive.

c) Solve: $z(xz_x - yz_y) = y^2 - x^2$. **(4+6+4)**

2.a) Find the orthogonal trajectories on the surface $x^2 + y^2 + 2fyz + d = 0$ of its curves of intersection with planes parallel to the $x - y$ plane.

b) Prove that a pfaffian differential equation $Pdx + Qdy + Rdz = 0$ is integnable if and only if $X.curl\,(X) = 0$, where $X = (P, Q, R)$. **(6+8)**

3.a) Find the integnal surface of the linear partial differential equation $x(y^2 + z)z_x - y(x^2 + z)z_y = (x^2 - y^2)z$ which contains the straight line $x + y = 0,\ z = 1$.

b) Find the general equation of surfaces orthogonal to the family given by $x(x^2 + y^2 + z^2) = C_1y^2$ showing that one such orthogonal set consists of the family of spheres given by $x^2 + y^2 + z^2 = C_2z$. **(7+7)**

4.a) Show that the equations $xz_x - yz_y = x$ and $x^2z_x + z_y = xz$ are compatible and find their solution.

b) Find the characteristics of the equation $z_xz_y = z$ and determine the integral surface which passes through the parabola $x = 0,\ y^2 = z$.

(7+7)

5.a) Find a complete integral of the partial differential equation $(p^2 + q^2)x = pz$ and deduce the solution which passes through the curve $x = 0,\ z^2 = 4y$.

b) Solve: $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$ **(8+6)**

Contd...2

6.a) Reduce the following equation to a connonical form and hence solve it:

$yz_{xx} + (x+y)z_{xy} + xz_{yy} = 0.$

b) Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin(x+2y)$ **(8+6)**

7.a) Solve the Dirichlet problem for a rectangle:

$z_{xx} + z_{yy} = 0,\ 0 \le x \le a,\ 0 \le y \le b$, subject to the BC's

$z(x,b) = z(a,y) = 0,\ z(0,y) = 0, z(x,0) = f(x).$

b) Obtain the D'Alembert's solution of the initial value problem of Cauchy type described as $Z_{tt} - C^2 Z_{xx} = 0, \quad -\infty < x < \infty,\ t > 0$

$ICs \quad z(x,0) = f(x), \quad Z_t(x,0) = g(x)$, where f and g are twice continuously differentiable functions on IR. **(8+6)**

8.a) A uniform rod of length L whose surface is thermally insulated initially at temperature $\theta = \theta_0$. At time $t = 0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature distribution $\theta(x,t)$.

b) A stretched string of finite length L is held fixed at its ends and is subjected to an initial displacement $u(x,0) = u_0 \sin\left(\frac{\pi x}{L}\right)$. The string is released from this position with zero initial velocity. Find the resultant time dependent motion of the string. **(7+7)**

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St Aloysius College (Autonomous)

Mangaluru

Semester IV – P.G. Examination - M. Sc. Mathematics

September - 2020

ALGEBRAIC NUMBER THEORY

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following (14x5=70)

1. a) Define Euler's totient function φ. Show that it is multiplicative.
   b) Prove or disprove: If $(m,n) = 1$, then $(\varphi(m), \varphi(n)) = 1$.
   c) If $(a,m) = 1$, then show that $a^{\varphi(m)} \equiv 1(mod\ m)$. Determine the last two digits of 3^{2020}.

(6+2+6)

2. a) Let $'p'$ be a prime and $f(x) = c_0 + c_1x + \cdots + c_nx^n$ be a polynomial with integer coefficients, such that $c_n \not\equiv 0(mod\ p)$. Then prove that the polynomial congruence $f(x) \equiv 0(mod\ p)$ has at most n solutions.
   b) State and prove Wilson's theorem.
   c) If p is an odd prime, then show that

   $$1^2.3^2.5^2 \ldots (p-2)^2 \equiv (-1)^{\frac{p+1}{2}}\ (mod\ p).$$

(5+5+4)

3. a) State and prove Gauss Lemma.
   b) If p and q are distinct odd primes, then prove that

   $$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{(p-1)(q-1)}{4}}$$

(7+7)

4. a) Prove or disprove: set of all algebraic numbers is uncountable.
   b) If α is a real algebraic number of degree $n > 1$, then show that there exists a positive constant $c(\alpha)$ such that for any rational number p/q with $(p,q) = 1, q > 0$, the inequality $\left|\alpha - p/q\right| > \frac{c(\alpha)}{q^n}$ holds.
   c) Show that $\sum_{n=0}^{\infty} \frac{1}{10^{n!}}$ is transcendental.

(4+5+5)

Contd...2

5. a) Let K be an algebraic number field and $[K : \mathbb{Q}] = n$. If $\alpha \in K$ and $\sigma_1, \sigma_2, \dots \sigma_n$ are the distinct $\mathbb{Q}$-isomorphisms of K into $\mathbb{C}$, then prove that

   i) $Tr_k(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \dots + \sigma_n(\alpha)$

   ii) $N_k(\alpha) = \sigma_1(\alpha)\,\sigma_2(\alpha) \dots \sigma_n(\alpha)$

   Further if $\alpha \in \mathcal{O}_k$, then show that $Tr_k(\alpha)$ and $N_k(\alpha)$ are integers.

   b) Prove that every algebraic number field has an integral basis.

(7+7)

6. a) If $K = \mathbb{Q}(\sqrt{d})$, where d is a square-free integer show that

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{d}, & if\ d \equiv 2\ or\ 3 (mod\ 4) \\ \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right), & if\ d \equiv 1 (mod\ 4) \end{cases}$$

   b) Solve $y^2 + 2 = x^3$, for $x, y \in \mathbb{Z}$.

(8+6)

7. a) Let K be an algebraic number field. If I, J are non-zero ideals of $\mathcal{O}_K$ with $I \subsetneq J$, then show that $N(I) > N(J)$.

   b) Define a Dedekind domain. Prove that $\mathcal{O}_K$ is a Dedekind domain.

(4+10)

8. a) If $\wp$ is a prime ideal of $\mathcal{O}_K$, then prove that $\wp^{-1}$ is a fractional ideal and $\wp\wp^{-1} = \mathcal{O}_K$.

   b) Prove that every ideal in $\mathcal{O}_K$ can be written as product of prime ideals uniquely.

(7+7)

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