PH 561.4

Reg. No. :

# St Aloysius College (Autonomous)

## Mangaluru

## Semester IV – P.G. Examination – M.Sc. MATHEMATICS

## August / September 2021

# MEASURE THEORY AND INTEGRATION

**Time: 3 Hours** **Max. Marks: 70**

**Answer any FIVE FULL questions from the following:** **(14x5=70)**

1.a) Define Lebesgue outer measure. Prove that Lebesgue outer measure is translation invariant.

b) Prove that outer measure of an interval equals its length.

**(4+10)**

2.a) Show that if $F$ is Lebesgue measurable and $m^*(F \Delta G) = 0$ then $G$ is Lebesgue measurable.

b) Let $\{E_i\}$ be a sequence of measurable sets. Prove the following:

i) if $E_1 \subseteq E_2 \subseteq \ldots$, then $m(\lim E_i) = \lim m(E_i)$

ii) if $E_1 \supseteq E_2 \supseteq \ldots$, and $m(E_i) < \infty, \forall i$, then $m(\lim E_i) = \lim m(E_i)$.

**(4+10)**

3.a) Let $c$ be any real number and let $f$ and $g$ be real valued measurable functions defined on the same measurable set $E$. Prove that $cf$, $f+g$, $fg$ are measureable.

b) Define a Lebesgue measurable function.

If $\{f_n\}$ is a sequence of Lebesgue measurable functions defined on the same measurable set, then show that $\inf f_n$ is Lebesgue measurable.

**(9+5)**

4.a) Define the Lebesgue integral of a non-negative measurable function.

b) If $f$ is Riemann integrable and bounded over the finite interval $[a,b]$ then prove that $f$ is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

**(2+12)**

5.a) State Fatou's lemma. State and prove the Lebesgue monotone convergence theorem.

b) If $f$ and $g$ are non-negative measurable functions and if $c$ is a non-negative real number, then show that $\int_a^b (cf+g)\,dx = c\int_a^b f\,dx + \int_a^b g\,dx$

**(5+9)**

Contd...2

6.a) Define a measure space. Show that $L^p(X,\mu)$ is a vector space over the real numbers for $1 \le p < \infty$.

b) Define a convex function on an interval (a,b) Prove that a convex function on (a, b) is continuous.

(5+9)

7.a) State and prove Holder's inequality.

b) Let $f$ and $g$ be non-negative measurable functions. Show that equality occurs in the Holder's inequality if and only if $sf^p + tg^q = 0$ $a.e$ for some constants $s$ and $t$, not both zero.

(7+7)

8. State and prove Hahn's lemma and Hahn decomposition theorem.

(14)

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# St Aloysius College (Autonomous)

## Mangaluru

### Semester IV – P.G. Examination – M.Sc. Mathematics

**August / September 2021**

## COMPLEX ANALYSIS II



**Time: 3 hrs.** **Max Marks: 70**

**Answer any FIVE FULL questions from the following:** **(14×5=70)**

1. a) Define a simply connected region. Give two examples.

   b) Prove that a region $\Omega$ is simply connected if and only if $n(\gamma, a) = 0$ for all cycles $\gamma$ in $\Omega$ and all points $'a'$ which do not belong to $\Omega$.

**(2+12)**

2. a) Evaluate $\int_c \frac{z^2+4}{z^2+2z^2+2z}\, dz$ by using residue theorem where $c: |z-1| = 5$.

   b) State and prove the Rouche's theorem.

   c) Evaluate $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\, dx$.

   d) Evaluate $\int_c \frac{e^z}{z^n}$, where c is the circle |z|=1.

**(5+5+2+2)**

3. a) State and prove residue theorem.

   b) Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\, dx, \quad a \in \mathbb{R}, \quad a \neq 0$.

   c) Define the residue of $f(z)$ at an isolated singularity 'a'.

**(7+5+2)**

4. a) State and prove the mean-value property for harmonic functions.

   b) If $u_1$ and $u_2$ are harmonic in a region $\Omega$, then prove that

   $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$ for every cycle $\Upsilon$ which is homologous to zero in $\Omega$.

**(7+7)**

5. a) Prove that a non-constant harmonic function defined in a region $\Omega$ has neither a maximum nor a minimum in $\Omega$.

   b) If $f(z)$ is analytic in $|z| < 1$ and satisfies $|f| = 1$ on $|z| = 1$, then show that $f(z)$ is rational.

   c) State Poisson's formula.

**(7+5+2)**

Contd...2

6.a) If $f(z)$ is analytic in an annulus $R_1 < |z-a| < R_2$ then show that $f(z)$ can be developed in a general power series of the form $f(z) = \sum_{n=-\infty}^{\infty} A_n \, (z-a)^n$ where

$$A_n = \frac{1}{2\pi i} \int_{|\tau-a|=r} \frac{f(\tau)}{(\tau-a)^{n+1}} d\tau \, .$$

b) State and prove Hurwitz theorem.

(7+7)

7. a) Suppose that $f_n(z)$ is analytic in a region $\Omega_n$ and that the sequence $\{f_n(z)\}$ converges to a limit function $f(z)$ in a region $\Omega$ uniformly on every compact subset of $\Omega$, then prove that $f(z)$ is analytic in $\Omega$.

b) If $f(z)$ is analytic in a region containing 'a' then show that the representation $f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n)}}{(n)!}(a)(z-a)^n + \cdots$ is valid in the largest open disc centered at '$a$' and contained in $\Omega$.

(7+7)

8. a) Prove that a necessary and sufficient condition for the absolute convergence of the product $\prod_{n=1}^{\infty}(1+a_n)$ is the convergence of the series $\sum_{n=1}^{\infty} |a_n|$.

b) Show that $\frac{\pi^2}{Sin^2 \pi z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$

c) Evaluate $\Gamma\left(\frac{1}{2}\right)$.

(7+5+2)

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# St Aloysius College (Autonomous)

## Mangaluru

## Semester IV – P.G. Examination –M.Sc. Mathematics

**August / September 2021**

## *FUNCTIONAL ANALYSIS*

Time : 3 hours

Max. Marks :70

**Answer any <u>FIVE</u> FULL questions from the following :**

1. a) State and prove the Cantor's intersection theorem.

   b) Prove that every complete metric space is of second category. (7+7)

2. Let $M$ be a closed linear subspace of a normed linear space $N$. Prove that $N/_M$ forms a normed linear space with respect to the norm given by $\|x+M\| = inf\{\|x+m\| : m \in M\}$, for every $x + M \in N/_M$. Show that if $N$ is a Banach space then so is $N/_M$. (14)

3. a) Let $N$ and $N'$ be normed linear spaces. Prove that the following are equivalent for a linear transformation $T: N \to N'$
   i) T is continuous
   ii) T is continuous at the origin
   iii) There exists a real number $k \geq 0$, such that $\|T(x)\| \leq k\|x\|$, for all $x \in N$.
   iv) If $S = \{x \in N: \|x\| \leq 1\}$ then $T(s)$ is bounded set in $N'$ .

   b) Let $N$ be a finite dimensional normed linear space with dimension $n > 0$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis of $N$. Show that the map $T: N \to l_1{}^n$ given by $T(x) = (x_1, x_2, \dots, x_n)$, whenever $x = x_1e_1 + x_2e_2 + \dots + x_ne_n$ uniquely, is continuous. (7+7)

4. a) Let $N$ be a nonzero normed linear space. Prove that $N$ is a Banach space if and only if $\{x \in N: \|x\| = 1\}$ is complete as a subspace of $N$.

   b) Define the conjugate space $N^*$ of a normed linear space $N$. Prove that there is an isometric isomorphism of $N$ into $N^{**}$. (7+7)

5. a) Let $B$ be a Banach space and $N$ be a normed linear space. Let $\{T_i\}_{i\in I}$ be a nonempty set of continuous linear transformation of $B$ in to $N$ such that $\{T_i(x)\}$ is a bounded subset of $N$, for each $x \in B$. Then prove that $\{T_i\}_{i\in I}$ is a bounded subset of $\mathcal{B}(B, N)$.

   b) If $B$ and $B'$ are Banach spaces and if $T$ is a continuous linear transformation of $B$ into $B'$, then show that the graph of $T$ is closed in $B \times B'$. (8+6)

6. a) Define a Hilbert space. Give an example.

   b) Prove that a closed convex set in a Hilbert space contains a unique vector of smallest norm.

   c) If $M$ is a proper closed linear subspace of a Hilbert space $H$, then prove that there exists a nonzero vector $z_0$ in $H$ such that $z_0 \perp M$. (2+6+6)

**Contd...2**

7. a) For an orthonormal set $\{e_1, e_2, \dots, e_n\}$ in a Hilbert space $H$ and $x \in H$ show that
   i) $\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \leq \|x\|^2$
   ii) $x - \sum_{i=1}^{n} \langle x, e_i\rangle e_i \perp e_j$, for each $j, 1 \leq j \leq n$.

   b) Let $H$ be a complex Banach space in which the parallelogram law holds. Then Prove that $H$ is a Hilbert space. (5+9)

8. a) Let $H$ be a Hilbert space, Prove that the adjoint operator $T \mapsto T^*$ on $\mathcal{B}(H)$ satisfies: i) $(T_1 + T_2)^* = T_1^* + T_2^*$
   ii) $\|T^*\| = \|T\|$

   b) Define a Unitary operator on a Hilbert space $H$. Show that a unitary operator $T$ is an isometric isomorphism of $H$ into itself. Is the converse true? Justify.

   c) Prove that an operator $T$on a Hilbert space is self adjoint if and only if $\langle Tx, x\rangle$ is real for all $x \in H$. (4+6+4)

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# St Aloysius College (Autonomous)

## Mangaluru

## Semester IV – P.G. Examination – M.Sc. MATHEMATICS

### August / September 2021

# PARTIAL DIFFERENTIAL EQUATIONS

**Time: 3 Hours** **Max. Marks: 70**

**Answer any FIVE FULL questions from the following:** **(14x5=70)**

1.a) Show that the general solution of Lagrange's equation $P(x,y,u)u_x + Q(x,y,u)u_y = R(x,y,u)$ is $F(\phi(x,y,u), \psi(x,y,u)) = 0$, where $F$ is an arbitrary function and $\phi(x,y,u) = C_1$ and $\psi(x,y,u) = C_2$ are integral curves of $\frac{dx}{P(x,y,u)} = \frac{dy}{Q(x,y,u)} = \frac{du}{R(x,y,u)}$.

b) Find the orthogonal trajectories on the conicoid $(x+y)z = 1$ of the conics in which it is cut by the planes $x - y + z = k$, where $k$ is a parameter.

**(7+7)**

2.a) Prove necessary and sufficient condition that there exists between two functions $u(x,y)$ and $v(x,y)$ a relation $F(u,v) = 0$, not involving $x$ or $y$ explicitly is $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

b) Test for integrability of $2xz(y-z)dx - z(x^2+2z)dy + y(x^2+2y)dz = 0$ and find its primitive.

**(7+7)**

3.a) Obtain the partial differential equation by eliminating the arbitrary function $f$ from $f\left(\frac{xy}{u}, \frac{x-y}{u}\right)$.

b) Find the integral surface of the partial differential equation $2y\,(u-3)u_x + (2x-u)\,u_y = y(2x-3)$ which passes through the circle $u = 0$, $x^2 + y^2 = 2x$.

c) Find the complete integral of $pqu = p^2(xp + p^2) + q^2\,(yp + q^2)$.

**(4+7+3)**

4.a) Find the characteristic of the equation $2pq - u = 0$ and determine the integral surface satisfying $u(0,y) = \frac{y^2}{2}$.

b) Derive a necessary and sufficient condition for the compatibility of $f(x,y,u,p,q) = 0$ and $g(x,y,u,p,q) = 0$.

**(8+6)**

5.a) Find a complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x = 0$ and $z^2 = 4y$.

b) Find the surface which intersects the system of surfaces $z(x+y) = c(3z+1)$ orthogonally and which passes through $x^2 + y^2 = 1$ and $z = 1$.

**(7+7)**

Contd...2

6.a) Solve $(D^3 + D^2)u = \sin(2x + y)\, e^{x-y}$.

b) Solve $(D^2 + 4DD' + 2D'^2) = \sin(x + 2y)$

c) Find the particular integral of the equation $(D - 3D' - 2)\, u = e^{2x}\, tan\,(3x + y)$.

**(5+5+4)**

7.a) Classify the equation $yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$ and reduce it to canonical form.

b) Find the characteristic curves of the partial differential equation $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$ and reduce it to canonical form.

**(8+6)**

8.a) Obtain the solution of the wave equation $u_{tt} = C^2 u_{xx}$ under the following conditions

i) $u(0, t) = u(2, t) = 0$

ii) $u(x, 0) = sin^3\left(\frac{\pi x}{2}\right)$

iii) $u_t(x, 0) = 0$

b) A uniform rod of length $L$ whose surface is thermally insulated initially at temperature $\theta = \theta_0$ at time $t = 0$, one end is suddenly pulled to $\theta = 0$ and subsequently maintained at this temperature, the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$.

**(6+8)**

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# St. Aloysius College (Autonomous), Mangaluru

## Semester IV – P.G. Examination – M. Sc. Mathematics

**August / September 2021**

## ALGEBRAIC NUMBER THEORY

Time : 3 Hours

Max. Marks : 70

**Answer any FIVE FULL questions:** $(14 \times 5 = 70)$

1. (a) If $n$ is a positive integer $> 1$ with $(n-1)! \equiv -1 \pmod{n}$, then show that $n$ is a prime.

   (b) Define a reduced residue system modulo $m$. Prove the Euler-Fermat theorem.

   (c) Let $p$ be a prime and let $f(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial with integer coefficients, where $c_n \not\equiv 0 \pmod{p}$. Prove that $f(x) \equiv 0 \pmod{p}$ has at most $n$ incongruent solutions mod $p$. **(3+5+6)**

2. (a) If $a$ is an integer and $p$ is an odd prime, then prove that $\left(\frac{a}{p}\right) = a^{\frac{p-1}{2}} \pmod{p}$.

   (b) For any odd prime $p$, evaluate $\left(\frac{2}{p}\right)$ and $\left(\frac{-1}{p}\right)$. **(7+7)**

3. (a) State and prove the Gauss' lemma.

   (b) Determine all odd primes $p$ such that 13 is a quadratic residue mod $p$. **(8+6)**

4. (a) Define an algebraic number. Show that the a rational number is an algebraic integer if and only if it is an integer.

   (b) Define the notions of trace and norm of an element in algebraic number field $K$. If $K$ is an algebraic number field of degree $n$ over $\mathbb{Q}$, then show that for any $\alpha \in K$,
   $$\mathrm{Tr}_K(\alpha) = \sigma_1(\alpha) + \sigma_2(\alpha) + \cdots + \sigma_n(\alpha), \text{ and } N_K(\alpha) = \sigma_1(\alpha)\sigma_2(\alpha)\cdots\sigma_n(\alpha),$$
   where $\sigma_1, \sigma_2, \ldots, \sigma_n$ are the $n$ distinct $\mathbb{Q}$-isomorphisms of $K$ into $\mathbb{C}$.

   (c) Determine the trace and norm of $i$ in $\mathbb{Q}(i)$. **(4+7+3)**

5. (a) Prove that every algebraic number field has an integral basis.

   (b) Prove that any two integral bases of an algebraic number field have the same discriminant.

   (c) Find the discriminant of $1+i$ in $\mathbb{Q}(i)$. **(7+4+3)**

6. (a) Let $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square-free integer. Show that the ring of algebraic integrs $O_K$ is
   $$\begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{d}, & \text{if } d \equiv 2 \text{ or } 3 \pmod{4}; \\ \mathbb{Z} + \mathbb{Z}\left(\frac{1+\sqrt{d}}{2}\right), & \text{if } d \equiv 1 \pmod{4}. \end{cases}$$



Contd...2

(b) Determine the discriminant of $K = \mathbb{Q}(\sqrt{d})$, where $d$ is a square-free integer. (10+4)

7. (a) Define a Dedekind domain. Prove that the ring of algebraic integers $\mathcal{O}_K$ of an algebraic number field $K$ is a Dedikind domain.

   (b) Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field $K$. Let $P$ be a non-zero prime ideal of $\mathcal{O}_K$ and let $P^{-1} = \{\alpha \in K : \alpha P \subseteq \mathcal{O}_K\}$. Show that $P^{-1}$ is a fractional ideal of $\mathcal{O}_K$, $\mathcal{O}_K \subsetneq P^{-1}$. (7+7)

8. Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field $K$.

   (a) Prove that every non-zero proper ideal of $\mathcal{O}_K$ contains a product of finitely many prime ideals of $\mathcal{O}_K$.

   (b) If $P_1, P_2, \ldots, P_r$ are distinct prime ideals of $\mathcal{O}_K$, $a_i \in \mathcal{O}_K$, $l_i \in \mathbb{N}$, $1 \leq i \leq r$, then show that there exists an $x \in \mathcal{O}_K$ such that $x \equiv a_i \pmod{P_i^{l_i}}$ for each $i$, $1 \leq i \leq r$. (5+9)

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