PH 561.1 **Reg. No:** | | | | | | |

St Aloysius College (Autonomous)

Mangaluru

Semester I – P.G. Examination- M.Sc. Mathematics

January - 2023

ALGEBRA I

Time: 3 Hours **Max. Marks: 70**

Answer any FIVE FULL questions from the following. **(14x5=70)**

1. a) Prove that a subset H of $\mathbb{Z}$ is a subgroup if and only if $H = a\mathbb{Z}$ for some $a \in \mathbb{Z}$.

   b) Let H and K be subgroups of a group G then show that HK is a subgroup of G if and only if $HK = KH$.

   c) Find all subgroups of the group $\mathbb{Z}_{24}$.

(8+4+2)

2. a) Prove that the group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if m and n are relatively prime.

   b) Prove that any subgroup of index 2 is normal.

   c) Let G be a group generated by an element a of order n then, prove that a^k is a generator of G if and only if $gcd(k,n) = 1$.

(6+3+5)

3. a) If G is a finite subgroup of the group O of rigid motions of the plane which fix the origin, then show that either G is the cyclic group of order n generated by the rotation ρ_θ, where $\theta = 2\pi/n$ or G is Dn, the dihedral group of order $2n$.

   b) Define isometry of $\mathbb{R}^2$. Prove that every isometry of $\mathbb{R}^2$ is a translation, a rotation or a reflection.

(7+7)

4. a) State and prove second Sylow theorem for a finite group.

   b) Prove that two elements in the symmetric group S_n are conjugates if and only if they have the same cycle structure.

(8+6)

5. a) State and prove Cayley's theorem.

   b) Derive the class equation for a finite group. Determine all possible class equations for a group of order 21.

(8+6)

6. a) If H and K are subgroups of group G, show that $[H : H \cap K] \leq [G : K]$.

   b) Show that every group of order 15 is simple.

   c) Find the class equation of S_3.

(5+6+3)

Contd...2

7\. a) Define an integral domain. Prove that every integral domain can be embedded in a field.

b) Prove that, an ideal M of a ring R is maximal ideal if and only if R/M is a field.

(8+6)

8\. a) Prove that every finite integral domain is a field.

b) State and prove the correspondence theorem on ring homomorphism.

c) Prove that an element of the ring $\mathbb{Z}_n$ is a divisor of zero if and only if it is not relatively prime to n.

(4+7+3)

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St Aloysius College (Autonomous)
Mangaluru
Semester I - P.G. Examinations - M.Sc. Mathematics
January 2023
LINEAR ALGEBRA-I

Time : 3 Hours **Max. Marks : 70**

Answer any FIVE FULL questions from the following: $(14 \times 5 = 70)$

1. (a) Define elementary matrices. Show that elementary matrices are invertible and their inverses are also elementary.

   (b) Prove that any $m \times n$ matrix can be reduced to a row-echelon form by applying finitely many elementary row operations.

   (c) Let A be an $n \times n$ matrix with integer entries. Prove that A is invertible, and that A^{-1} has integer entries if and only if $\det A = \pm 1$. (4+6+4)

2. (a) Prove that the following conditions are equivalent for a square matrix A

   (i) A can be reduced to the identity matrix by applying elementary operations

   (ii) A is a product of elementary matrices

   (iii) A is invertible

   (iv) the system $AX = 0$ has only the trivial solution.

   (b) Prove that a square matrix is invertible if and only if its determinant is nonzero.

   (c) Show that the inverse of a permutation matrix is its transpose.

   (d) Write the permutation matrix associated with the permutation $p = (1\ 4\ 3) \in S_5$. (6+4+2+2)

3. (a) Define a basis for a vector space. Verify whether $\mathcal{B} = \{(5,6,3)^t, (1,0,4)^t, (3,1,6)^t\}$ is a basis of $\mathbb{R}^3$ over $\mathbb{R}$.

   (b) Let V be a vector space over a field F. If S is a finite set of vectors in V which spans V, then show that S contains a basis of V.

   (c) If S and L are finite subsets of a vector space V over a field F such that S spans V and L is linearly independent, then prove that $|L| \leq |S|$. Deduce that any two bases of a finite dimensional vector space have the same number of elements. (3+3+8)

4. (a) Let V be an n-dimensional vector space and let $\mathcal{B}$ be an ordered basis of V. Prove that the collection of all ordered bases of V is $\{\mathcal{B}P : P \in GL_n(F)\}$.

   (b) Determine the order of $GL_n(F_p)$, where p is a prime.

   (c) Find a basis for the space of all solutions of the homogeneous system of equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$ (6+4+4)

5. (a) Show that the space $M_n(\mathbb{R})$ of all $n \times n$ real matrices is a direct sum of the space of all $n \times n$ real symmetric and the space of all $n \times n$ real skew-symmetric matrices.

Contd...2

(b) If W_1 and W_2 are subspaces of a finite dimensional vector space over a field, then prove that $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$. (5+9)

6. (a) State and prove the rank-nullity theorem for a linear transformation on a finite-dimensional vector space.

(b) Prove that similar matrices have same eigenvalues.

(c) Diagonalize the matrix $\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$. (6+3+5)

7. (a) Let T be a linear map of the vector spaces V and W over a field F, of dimensions n and m respectively. Prove that there exist bases $\mathcal{B}$ and $\mathcal{C}$ of V and W respectively, such that the matrix of T with respect to $\mathcal{B}$ and $\mathcal{C}$ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, where $r = \operatorname{rank} T$.

(b) Let $T : V \to V$ be a linear operator on a finite dimensional vector space V over a field F. Prove that the following are equivalent:

(i) $\ker T > 0$

(ii) $\operatorname{im} T < V$

(iii) If A is the matrix of T with respect to an arbitrary basis, then $\det A = 0$.

(iv) 0 is an eigenvalue of T. (7+7)

8. (a) Prove that an $n \times n$ real matrix A is orthogonal if and only if $(AX \cdot AY) = (X \cdot Y)$ for all $X, Y \in \mathbb{R}^n$.

(b) Show that a rigid motion of $\mathbb{R}^n$ is the composition of an orthogonal linear operator and a translation. (4+10)

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St Aloysius College (Autonomous)

Mangaluru

Semester I– P.G. Examination – M.Sc. Mathematics

January - 2023

REAL ANALYSIS – I

Time: 3 hrs. **Max Marks: 70**

Answer any FIVE FULL questions. **(14x5=70)**

1\. a) Define the notions of an upper bound and least upper bound of a subset E of an ordered set S.

b) State and prove the Archimedean property of $\mathbb{R}$.

c) State and prove the Schwarz inequality.

d) Let A be a countable set and let B be the set of all n-tuples of elements of A. Prove that B is countable. **(2+4+4+4)**

2\. a) In a metric space prove that any neighbourhood of a point is an open set.

b) Let X be a metric space and $E \subseteq X$. Then prove that E is open if and only if E^C is closed.

c) Prove that a finite union of closed sets in a metric space is a closed set. Is an arbitrary union of closed sets closed? Justify your answer. **(4+6+4)**

3\. a) If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is non empty, then prove that $\cap K_\alpha$ is nonempty.

b) Define a compact set. Prove that every $k-cell$ in $\mathbb{R}^k$ is compact. **(4+10)**

4\. a) Define a Perfect Set. If P is a nonempty perfect set in $\mathbb{R}^k$, prove that P is uncountable.

b) Define a connected set. Prove that a subset E of $\mathbb{R}$ is connected if and only if it is an interval. **(6+8)**

5\. a) If $\{s_n\}$ and $\{t_n\}$ are complex sequences, such that $\lim_{n\to\infty} s_n = s$, and $\lim_{n\to\infty} t_n = t$, prove that $\lim_{n\to\infty} s_n t_n = st$.

b) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$ for real $p > 0$ and real α.

c) Show that the set of all subsequential limits of a sequence $\{p_n\}$ in a metric space X forms a closed subset of X. **(3+7+4)**

6\. a) State and prove Ratio test for convergence of a series.

b) For any sequence $\{c_n\}$ of positive real numbers, prove that

$$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \le \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}.$$

c) Prove that 'e' is irrational. **(6+4+4)**

Contd...2

7. a) Let X and Y be metric spaces and $f: X \to Y$ be a function. Prove that f is continuous if and only if for every sequence $\{x_n\}$ converging to x in X, the sequence $\{f(x_n)\}$ converges to $f(x)$ in Y.

b) Define uniform continuity. Prove that a continuous mapping of a compact set X into a metric space Y is uniformly continuous.

c) Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if $'c'$ is a number such that $f(a) < c < f(b)$, then prove that there exists a point $x \in (a, b)$ such that $f(x) = c$. **(4+6+4)**

8. a) Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f and g is differentiable at the point $f(x)$. If $h(t) = g(f(t))$, $a \le t \le b$. Then prove that h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$.

b) State and prove Taylor's theorem.

c) Suppose f is defined in a neighbourhood of x, and suppose $f''(x)$ exists. Show that $\lim_{h \to 0} \frac{f(x+h) - f(x-h) - 2f(x)}{h^2} = f''(x)$. **(5+6+3)**

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St Aloysius College (Autonomous)
Mangaluru
Semester I - P.G. Examinations - M.Sc. Mathematics
January 2023
GRAPH THEORY

Time : 3 Hours **Max. Marks : 70**

Answer any FIVE FULL questions from the following: **($14 \times 5 = 70$)**

1. (a) Define the intersection number $\omega(G)$ of a graph G. Let G be a connected graph with $p > 3$ points. If G has no triangles, then show that $\omega(G) = q$, the number of lines of G.

   (b) Prove that a graph is bipartite if and only if all its cycles are even. (7+7)

2. (a) Show that the maximum number of lines among all p point graphs with no triangles is $\left[\frac{p^2}{4}\right]$.

   (b) Prove or disprove the following for any two graphs G_1 and G_2:

   (i) If G_1 and G_2 are regular then so is $G_1 + G_2$.

   (ii) $\overline{G_1 + G_2} = \overline{G_1} + \overline{G_2}$.

   (10+4)

3. (a) If x is a line of a connected graph G prove that the following are equivalent:

   (i) x is a bridge of G.

   (ii) x is not on any cycle of G.

   (iii) There exist points u and v of G such that the line x is on every $u - v$ path.

   (iv) There exists a partition of V into subsets U and W such that for every $u \in U$ and $w \in W$, the line x is on every path joining u and w.

   (b) Prove that every nontrivial connected graph has at least two points which are not cut points.

   (8+6)

4. (a) Prove that the following statements are equivalent for a (p, q) graph G:

   (i) G is a tree.

   (ii) Every two points of G are joined by a unique path.

   (iii) G is connected and $p = q + 1$.

   (iv) G is acyclic and $p = q + 1$.

   (b) Define center of a graph G. Prove that every tree has a center consisting of either one point or two adjacent points.

   (9+5)

5. (a) For any graph G, show that $\kappa(G) \leq \lambda(G)$.

   (b) Prove that there exists a system of distinct representatives for a family of sets $S_1, S_2, \ldots, S_m$ if and only if the union of any k of these sets contain at least k elements, for all k, $1 \leq k \leq m$.

   (5+9)

contd...2

6. Let G be a graph having $p \geq 3$ points. If for every n, $1 \leq n \leq \frac{p-1}{2}$ the number of points of degree not exceeding n is less than n and if, for odd p, the number of points of degree at most $\frac{p-1}{2}$ does not exceed $\frac{p-1}{2}$, then prove that G is Hamiltonian. (14)

7. (a) Define an Eulerian and a Hamiltonian graph. Give an example of a graph which is Eulerian but not Hamiltonian.

   (b) Prove that every planar graph G with $p \geq 4$ points has at least four points of degree not exceeding 5

   (c) Define a plane map. For a plane map with vertices, edges and faces, prove that $p - q + r = 2$. (3+6+5)

8. (a) For any graph G, show that $\chi(G) \leq 1 + \max \delta(G')$ where the maximum is taken over all induced subgraph G' of G and $\delta(G')$ is the minimum degree of G'.

   (b) Prove that every planar graph is 5-colorable. (5+9)

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PS 567.1 Reg No.

St Aloysius College (Autonomous)

Mangaluru

Semester I – P.G. Examination –M.Sc. Mathematics

January - 2023

ORDINARY DIFFERENTIAL EQUATIONS

Time : 3 hours **Max. Marks :70**

Answer any FIVE FULL questions from the following:

1. a) Prove or disprove the following:

   i) The functions $x_1(t), x_2(t), \ldots\ldots\ldots, x_n(t)$ are linearly independent on an interval I of $\mathbb{R}$ if $W\big(x_1(t_0),\ x_2(t_0), \ldots\ldots, x_n(t_0)\big) \neq 0$ for some $t_0 \in I$.

   ii) If the functions $x_1(t), x_2(t), \ldots\ldots\ldots, x_n(t)$ are linearly independent on an interval I of $\mathbb{R}$ then $W\big(x_1(t_0),\ x_2(t_0), \ldots\ldots, x_n(t_0)\big) \neq 0$ for some $t_0 \in I$.

   b) State and prove Abel's formula for n^{th} order linear homogeneous differential equation.

   c) Verify if $e^x, \cos x, \sin x$ are linearly independent on the real line.

(6+6+2)

2. a) Let φ_1 be a solution of $L_2(x) = a_0(t)x'' + a_1(t)x' + a_2(t)x = 0,\ t \in I$ where $a_0(t) \neq 0, \forall\, t \in I$ and $\varphi_1(t) \neq 0, \forall\, t \in I$. Then show that

$$\varphi_2(t) = \varphi_1(t) \int_{t_0}^{t} \frac{1}{\big(\varphi_1(s)\big)^2} \exp\left[-\int_{t_0}^{s} \frac{a_1(\xi)}{a_0(\xi)}\, d\xi\right] ds$$

   is a solution of the above equation on I and $t_0 \in I$. Further show that $\varphi_1(t)$ and $\varphi_2(t)$ are linearly independent on I.

   b) Find the general solution of $t^2x'' - 2tx' + 2x = t^3 \sin t\,, t \in (0, \infty)$

(7+7)

3. a) Solve the IVP

$$x''' + x'' = 0, x(0) = 1, x'(0) = 0, x''(0) = 1$$

   b) Solve $x^{(4)} + 4x = 2\cos t + 1 + t + t^2 + e^{5t}$. **(5+9)**

4. a) State and prove the Rodrigues formula for Legendre polynomials.

   b) State and prove the orthogonal property of the Legendre polynomials. **(6+8)**

5. a) Derive Bessel's function of the first kind.

b) With usual notations for Bessel functions show that:

i) $\frac{d}{dt}\left(t^{p}J_{p}(t)\right)=t^{p}J_{p-1}(t)$

ii) $\frac{d}{dt}\left(t^{-p}J_{p}(t)\right)=-t^{-p}J_{p+1}(t)$. **(8+6)**

6. a) Solve the given system of linear equations

$$x' = 4x - y, \qquad y' = 2x + y$$

b) Prove that the set of all solutions of the system $X'(t) = A(t)X, t \in I$ forms an n −dimensional vector space over the field of complex numbers.

c) Let Φ be a fundamental matrix of the system $X'(t) = A(t)X, t \in I$ and let C be a constant, non-singular matrix. Then show that ΦC is also a fundamental matrix of the given system. Further prove that every fundamental matrix of the given system is of this type for some constant, non-singular matrix C.

(3+6+5)

7. a) Show that $\Phi(t) = \begin{bmatrix} e^{3t} & 2te^{3t} \\ 0 & e^{3t} \end{bmatrix}$ is a fundamental matrix of the system $X'(t) = A(t)X$ where $A(t) = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$.

Further find the solution of $X'(t) = A(t)X(t) + B(t)$ satisfying the condition $X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, where $B(t) = \begin{bmatrix} e^{t} \\ e^{-t} \end{bmatrix}$

b) Let $A(t)$ be an $n \times n$ continuous matrix on I and be periodic with period ω. If $\Phi(t)$ is a fundamental matrix for the system $x'(t) = A(t)\,x$, then show that $\Phi(t+\omega)$ is also a fundamental matrix. Justify that for any such $\Phi(t)$, there exists a periodic non-singular $P(t)$ with period ω and a constant matrix R such that $\Phi(t) = P(t)\,e^{tR}$. **(8+6)**

8. State and prove Picard's theorem. **(14)**

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